

Demand Elasticity in Dynamic Asset Pricing*

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Abstract

The demand-system approach identifies asset demand slopes using residual supply shocks, presuming these shocks move expected returns but not risk. In dynamic economies, risk is endogenous: investors retrade, so demand depends on the joint distribution of current and future returns, which changes as investors absorb a shock. The shock therefore changes the demand curve it traces, violating the exclusion restriction. In a calibrated multi-asset dynamic model, the measured slope is roughly 40% of its conceptual counterpart, implying substantially steeper demand curves than standard estimates suggest. The bias persists even if shocks are infinitesimal and transitory. We discuss potential empirical remedies.

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1 Introduction

Investors form portfolios by trading off expected returns against risk; their demand function maps measures of expected returns and risk into portfolio weights. The demand-system approach to asset pricing, initiated by the seminal work of Koijen and Yogo (2019), aims to estimate this function from data. The identification strategy is to find (residual) supply shocks that move expected returns while leaving the other determinants of demand unaffected, so that the resulting change in equilibrium quantities reveals the demand response. Implicit in this strategy is a question that must be answered first: what observable object is a sufficient statistic for “expected returns,” and what is a sufficient statistic for “risk”?

In a static economy, the answer is simple. Investors hold a portfolio and consume its dividends, so the current price vector summarizes expected returns, and risk is the exogenous joint distribution of dividends. The linearized demand function is then captured by an elasticity matrix, with each element measuring the sensitivity of demand for asset i to the price of asset j . This matrix is identified from shocks to the equilibrium price vector, with observable asset characteristics controlling for risk.¹ In a dynamic economy, this picture breaks down. Investors retrade, so what matters for portfolio choice is the joint distribution of current *and* future prices. Two consequences follow. First, the current price vector is no longer a sufficient statistic for expected returns, which now depend on future prices as well. Second, risk emerges from the same equilibrium that determines those future prices, so observable characteristics no longer suffice to summarize it. Each consequence opens a distinct margin on which the standard identification strategy can break. While the existing literature has offered some analysis on the first margin,² this paper takes up the second: we ask how demand-curve estimation is affected when the shocks used to identify the response to expected returns also move risk in equilibrium.

The identification strategy rests on an exclusion restriction: the supply shock must move expected returns and nothing else that demand depends on. In a dynamic economy the restriction fails even when the shock is orthogonal to asset fundamentals. Due to the shock, investors must hold a different quantity of risk in equilibrium, and that change reshapes the joint distribution of returns itself. The shock therefore moves not only expected returns—the argument the econometrician seeks to vary—but also the risk arguments the standard procedure implicitly holds fixed. The exclusion restriction is violated, and the apparent relationship between demand and expected returns both shifts and tilts.³

¹Estimating the full elasticity matrix without further restrictions is generally infeasible. Fuchs, Fukuda and Neuhaan (2025) discuss why focusing on the diagonal of the matrix can be problematic, and Haddad, He, Huebner, Kondor and Loualiche (2025) provide a systematic approach to estimating the full matrix by imposing restrictions motivated by finance theory.

²Termed the “pass-through” from the price level to expected returns, it is the focus of an emerging literature (e.g. Gabaix and Koijen, 2022; Davis, Kargar and Li, 2023; van Binsbergen, David and Opp, 2025).

³Recent empirical work is consistent with this insight. Corell, Holm-Hadulla, Leombroni, Mota and Pa-

To sign and quantify this mismeasurement, we develop and calibrate a dynamic model in which investors solve a Merton-style portfolio problem. Their demand depends on the joint distribution of returns through three sufficient statistics: the vector of instantaneous expected returns, the covariance matrix of instantaneous returns, and the covariance of current returns with future investment opportunities (Merton, 1971). The supply shock moves all three. We compare two objects: the *measured slope*, the equilibrium change in investors’ demand per unit change in expected returns induced by the supply shock; and the *conceptual slope*, the optimal quantity response of a representative investor per unit change in instantaneous expected returns, derived from her first-order condition. In our calibrated examples, the measured slope is approximately 40% of the conceptual slope: demand curves are substantially steeper than standard estimates suggest. The distortion arises through two channels, corresponding to the second and third components of Merton’s representation. An *endogenous risk effect* operates through the covariance matrix, as supply shocks change return volatilities and covariances. An *amplified intertemporal hedging effect* operates through the third component, as the shocks change how returns comove with future investment opportunities.

Our model is a continuous-time, multi-asset, multi-agent asset pricing economy. Assets are claims on risky cash flows driven by a multidimensional Brownian shock. Two types of agents populate the economy: hedgers and investors. Investors are long-lived, have power utility, and solve a canonical consumption-savings problem with optimal portfolio decisions as in Merton (1971). For simplicity, hedgers are short-lived and act as residual demand shifters. They receive a stochastic endowment correlated with the assets’ cash flows, with the correlation switching between a “normal” and a “stressed” state according to a Poisson process. In the stressed state, hedgers reduce their demand for a subset of assets, leaving investors to absorb the residual supply. These supply shocks drive our analysis. The intensity of the Poisson process determines their persistence, and equilibrium asset prices adjust endogenously.

Consider a single risky asset. Can the price and quantity responses to an exogenous shift in residual supply recover the slope of investors’ demand? In a static version of our economy, the answer is affirmative, consistent with the static-economy argument above. As in the Markowitz model, the slope of investors’ static demand is the ratio of risk tolerance to return variance, both primitives that the supply shock leaves untouched. When hedgers demand less of the asset, investors must hold more, raising its expected equilibrium return. Because the demand curve does not move, comparing the change in investors’ demand to the induced change in expected returns recovers its slope.

The dynamic economy behaves differently, even in its simplest specification, where in-

poutsi (2025) argue that the ECB’s quantitative easing program shifted the demand curve of mutual funds by changing the price at which funds expect to liquidate their positions, and Haddad, Moreira and Muir (2024) show that the Federal Reserve’s quantitative easing program changed market perceptions of the future return distribution of government bonds, thereby shifting investors’ demand for them.

vestors have log utility. As in Merton (1971), the slope of a log investor’s demand curve is the inverse of return variance. However, return volatility is endogenous in a dynamic economy. When hedgers partially liquidate their positions following the shock, investors must absorb more risk in equilibrium, raising the volatility of the asset’s return. This is a standard amplification mechanism in heterogeneous-agent models: investors’ larger risk exposure means that adverse future shocks reduce their risk-bearing capacity, raising the risk premium and deepening the price drop, which amplifies the initial effect.⁴ Because the slope of the demand curve is inversely related to return volatility, shifts in residual supply flatten the slope we seek to recover. The tractability of the log-investor framework allows us to show formally that the measured demand curve is flatter than its conceptual counterpart.

A natural conjecture is that empiricists can escape this distortion by relying on short-lived or small supply shocks. We show the conjecture fails in both limits, for distinct reasons. When the shock is fully transitory, investors’ equilibrium positions still differ from their pre-shock positions during the instant the shock lasts, so endogenous volatility differs at precisely the moment the slope is measured. When the shock is diminishingly small, the change in holdings (the numerator of the slope) and the change in expected returns (the denominator) both shrink toward zero, but proportionally; the ratio that defines the slope remains bounded away from its conceptual counterpart. This remains the case even in the joint limit—that is, when the shock is both infinitesimal and fully transitory.

Our second specification turns to multiple assets, partitioned into two groups—blue and red. All assets have the same cash-flow risk, but hedgers demand more of the blue assets, a pattern that can represent mandates or index composition. We specify the normal and stressed states so that the transition between them amounts to relabeling a single blue asset as red and a single red asset as blue, mimicking an index-inclusion event. In the stressed state, hedgers cut their positions in the newly red asset and investors absorb the residual supply. For that asset, both its return volatility and its covariance with every other asset rise. The intuition mirrors the single-asset case. Investors, by construction, carry larger exposure to the dividend risk of red assets. When future cash flows of red assets are low, investors’ aggregate wealth falls more, reducing their risk-bearing capacity. This raises the risk premia of all assets, but the price drop is disproportionate for red assets, which investors must continue to hold in greater quantity. Red assets are therefore more volatile and more correlated with investors’ aggregate wealth.⁵

In our multi-asset economy, the slope of investors’ demand curve is proportional to the

⁴Larger risk exposure leading to larger endogenous volatility is a standard effect in dynamic asset pricing models with heterogeneous agents, including Longstaff and Wang (2012), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), and Kondor and Vayanos (2019), among others. It is also closely related to the well-established empirical regularity of the leverage effect (Black, 1976).

⁵The same mechanism arises in cross-sectional empirical settings: comparing a blue and a red asset in cross-section or following a single asset across the shock yields close to identical results; see Section 4.3.

inverse of the covariance matrix, so the variance and covariance channels work in the same direction. Both tilt the demand curve, making the measured slope flatter than its conceptual counterpart. We call this distortion, operating through endogenous volatilities and covariances, the *endogenous risk effect*. As in the single-asset case, this effect tends to be negative, and does not vanish even when the shock to relative supply between assets is diminishingly small.

When investors’ relative risk aversion differs from one, a further effect emerges. Investors prepare for future states in which others are forced to scale back. The assets that matter most in those states are those held most heavily by other investors: an adverse cash-flow shock to such an asset erodes their wealth, forces them to deleverage, and sharply raises its risk premium. As a result, each investor’s hedging demand depends on other investors’ holdings. Supply shocks that shift hedgers’ positions onto investors change aggregate holdings and thus amplify this dependence. We call this general-equilibrium amplification the *amplified intertemporal hedging effect*. It appears in our model in two forms, distinguished by the risks investors hedge against: the Brownian innovation to future cash flows, or the eventual reversal of the supply shock itself.

We then quantify these economic effects through two calibrated applications. The first targets the factor-elasticity context, calibrated to Ben-David, Li, Rossi and Song (2021), who measure the price effect of a persistent change in mutual-fund demand following a change in Morningstar’s rating methodology. The second targets the micro-elasticity context, gauging the likely magnitude of “index inclusion” (Shleifer, 1986; Greenwood and Sammon, 2024).

Consistent with Gabaix and Koijen (2022), the measured slope differs in magnitude between the factor-elasticity and micro-elasticity exercises, but the pattern of mismeasurement is similar across both calibrations. First, in each calibration the measured slope is roughly 40% of its conceptual counterpart, suggesting that investors’ demand curves may be considerably steeper than standard estimates imply. Second, our decomposition attributes the bulk of the deviation to the endogenous risk effect. The amplified intertemporal hedging effect accounts for the remainder, almost entirely through its Brownian-shock form: this form works against the endogenous risk effect when investors are more risk-averse than the log investor and reinforces it otherwise, and can account for up to 30% of the mismeasurement when the implied return shock is large and persistent. The reversal-hedging form is small.

We close our analysis by asking what the way forward looks like. Once the supply shock moves all three components of Merton’s representation, four routes are worth considering. Focusing on small or transitory shocks reduces the mismeasurement, but our calibrated limit result shows it remains sizable. Adding controls for risk targets the violated exclusion restriction directly, but the controls are themselves moved by the shock, creating a bad-control problem (Angrist and Pischke, 2009). A more promising avenue is to take seriously the structure of the demand curve implied by the model in hand and find a separate instrument for

each of its arguments—that is, each Merton component in our model.⁶ Finally, one may step back from estimating the investor’s demand function altogether and treat the price response to exogenous supply shocks, often called the price multiplier, as the main object of interest. The drawback is that dropping the structural interpretation likely reduces external validity: each context yields its own estimate.⁷

Literature review. To our knowledge, this paper is the first to highlight that exogenous shocks to residual supply—arising from dynamic general equilibrium effects through changing risk—can alter the very demand slope that econometricians seek to estimate. Building on this intuition, our paper connects and extends several strands of the existing literature.

The seminal papers on demand-system estimation, (Kojen and Yogo, 2019; Kojen, Richmond and Yogo, 2023; Gabaix and Kojen, 2022) set up dynamic models as the conceptual background for their estimates. However, in these frameworks the possibility of our effects is ruled out by assumption. For instance, in Kojen and Yogo (2019) the form of the covariance matrix is determined by an assumption under which it is independent of the residual supply shocks. This rules out the endogenous risk effect. Similarly, the main model in Gabaix and Kojen (2022) (as does its generalized version in Gabaix, Kojen, Mainardi, Oh and Yogo (2025)) assumes that investors’ response to changes in expected returns is driven by an exogenous, constant coefficient. In contrast, our model allows for endogenous risk deriving the equilibrium covariance matrix from first principles.

The recent papers of An (2024), Fuchs et al. (2025) and Haddad, He, Huebner, Kondor and Loualiche (2025) (HHHKL, from now on) focus on problems stemming from the fact that investors’ demand curve for a given asset potentially depends on the prices of all assets in the investment opportunity set. HHHKL work out a flexible method to estimate each coefficient of the resulting demand curve, using the cross-section for relative elasticity between assets with the same observables and the time-series for substitution across assets with different observables.⁸ As these papers focus on a static economy where demand curve coefficients are exogenously given by primitives, while we allow the econometrician to observe the effect of the shock to all prices, our work is orthogonal to this literature. That is, all our effects are above and beyond the ones highlighted in these papers.

The concurrent work by van Binsbergen et al. (2025) also highlights that supply shocks used as instruments can violate the exclusion restriction in a dynamic economy. Their focus

⁶Davis, Kargar, Li and Silva (2026) take a first step in this direction; see the discussion in Section 6.3.

⁷This trade-off is familiar from the literature on the government spending multiplier: the multiplier is likewise not a structural parameter, as its magnitude changes with the nature of the policy, yet it has proven valuable as an identified moment disciplining structural models (Ramey, 2019; Nakamura and Steinsson, 2018). Section 6.3 develops this parallel.

⁸In Appendix B, we show that the investors’ endogenous slope matrix derived in our fully dynamic model locally satisfies the restriction imposed in HHHKL.

is the first margin we identified above: they study the mapping between an instrumented price-level shift and the corresponding shift in expected returns. They establish sufficient conditions under which this mapping is clean—the price shift must resolve within a single trading period and be unanticipated—and characterize the bias in price-elasticity estimates when those conditions fail.⁹ Our work takes up the second margin: we work directly with expected-return variation and ask whether the supply shock simultaneously moves the risk arguments of the demand function. The two analyses are complementary, and our mechanism applies whether the econometrician estimates return elasticity or price elasticity; Section 6.2 formalizes this point.

As we noted, the endogenous risk effect is closely connected to the well-established “leverage effect” – the empirical regularity that volatility rises when stock prices fall (Black, 1976; Christie, 1982). Intertemporal hedging demand, on the other hand, is an important ingredient for understanding asset pricing (Campbell, Giglio, Polk and Turley, 2018; Campbell, Giglio and Polk, 2023). On the modeling side, the endogenous risk effect arises in a wide range of heterogeneous-agent models, with or without explicit financial frictions (e.g. Longstaff and Wang, 2012; He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014; Kondor and Vayanos, 2019); we build on Kondor and Vayanos (2019) for its simplicity and its ability to accommodate an N -asset structure. None of these papers address the problem of recovering demand sensitivities using exogenous residual supply shocks, and we are unaware of any other models that emphasize the amplified intertemporal hedging effect.

Finally, in the companion paper He, Kondor and Li (2026), we take a step back from the view that asset markets can be seen as a demand system with well-defined supply and demand shocks. Instead, we focus on the quantitative effect of an exogenous supply change on equilibrium returns and prices.¹⁰ As we explain in this paper, this shock affects the demand curves of each group, hence the multiplier does not correspond to the inverse of the demand elasticity for any group. Still, we provide sophisticated benchmarks for the literature on what “canonical” models imply on this aggregate multiplier and how sensitive it is to changes of parameters in the economy.

⁹As a connected empirical point, Davis et al. (2023) argue that price elasticity estimates are sensitive to the “price pass-through”—a concept closely related to the pace of resolution of the price shock— which strongly influences the mapping between price and return elasticity measures. See also An and Zheng (2025) for a dynamic asset pricing model with a role for the predictability of flow shocks.

¹⁰In this regard our paper is also related to Berk and Van Binsbergen (2025), who show that the magnitude of the index inclusion effect in recent years (Greenwood and Sammon, 2024) is consistent with a classic CAPM setting, once carefully calibrated to take into account the wealth share of index investors and the persistence of their demand shift due to index inclusion. We offer a more detailed discussion on this issue in Section 6.1.

2 The Tilt and the Shift: A Minimal Demand System

To set the stage, we start with an illustrative example. We present a demand system with the minimal reduced-form properties required for our main effects to be present. In the dynamic model, these properties arise endogenously.

Minimal demand system. Consider the following minimal demand system with a single asset, which is given as follows:

$$\begin{aligned} \text{hedger's demand: } \hat{x} &= c [\mathbb{E}(R) - r] + \varepsilon, \\ \text{investor's demand: } \hat{y} &= a^\xi + b^\xi [\mathbb{E}(R) - r], \\ \text{market clearing: } S &= x + y. \end{aligned} \tag{1}$$

Here, r denotes the exogenous risk-free rate while $\mathbb{E}(R)$ denotes the asset's *endogenous* expected return. The individual demand of any representative hedger and investor, denoted by $\hat{x}$ and $\hat{y}$ respectively, depends on the expected excess return $\mathbb{E}(R) - r$ with ε being an orthogonal demand shock. We will consider below the effect of two realizations of the shock to hedgers' demand, $\varepsilon = \{\varepsilon^\xi\}$ with $\xi = \{s, n\}$ for stressed and normal. When $\varepsilon^s < \varepsilon^n$ hedgers dump assets on investors in the stressed state. The positive constant S is the asset's aggregate supply, which is fixed in our asset pricing context. As x and y are the aggregate counterparts of demand, the last equation represents market clearing.¹¹

The only non-standard element of the minimal asset demand system (1) is on the investor's side: both the slope, b^ξ , and the intercept, a^ξ are indexed by the state $\xi = \{s, n\}$. That is, we allow these parameters to change with the demand shock.

Demand slope and measured slope. We think of the slope of an agent's demand as her optimal quantity response to unit change in expected returns. Our focus is on the slope of the investors' asset demand. Suppose that an investor is asked how much she would increase her demand for the asset if $\mathbb{E}(R)$ would increase to $\mathbb{E}(R) + \delta$ where δ is an arbitrary constant. In the demand system in (1), this gives the *conceptual* slope of investors' demand curve in state $\xi = n$ (we will use $\mathcal{C}$ for conceptual):

$$\mathcal{C}^{\min} \equiv \frac{\hat{y}^h - \hat{y}}{\delta} = b^n, \tag{2}$$

where $\hat{y}^h$ is the demand under the *hypothetical* increase in expected excess return.

¹¹For simplicity, in this minimal example demands, x and y are in units of assets. As we will see when we specify our model, the natural units depend on the utility functions of the agents; Section 6.2 shows that our main point does not depend on this choice.

Now suppose instead that we observe the equilibrium under the two realizations $\varepsilon = \{\varepsilon^n, \varepsilon^s\}$. By only shocking the x -demand-curve, we might be able to trace out the slope of y -demand-curve. In fact, the measured equilibrium ratio of changes in quantities and returns, that is, the *measured slope* is (we will use $\mathcal{M}$ for *measured*)

$$\mathcal{M}^{\min} \equiv \frac{y^s - y^n}{\mathbb{E}(R^s) - \mathbb{E}(R^n)} = \underbrace{b^n}_{\text{demand slope in } \xi = n} + \underbrace{\frac{(b^s - b^n)[\mathbb{E}(R^s) - r]}{\mathbb{E}(R^s) - \mathbb{E}(R^n)}}_{\text{endogenous risk effect}} + \underbrace{\frac{a^s - a^n}{\mathbb{E}(R^s) - \mathbb{E}(R^n)}}_{\text{amplified hedging effects}}. \quad (3)$$

Clearly, if the slope and the intercept of investors' demand do not respond to the shock ε , that is, $b^s = b^n$ and $a^s = a^n$, this strategy identifies slope b^n . This is the textbook case.

However, if the demand curve tilts, $b^s \neq b^n$, or shifts, $a^s \neq a^n$, the second and third terms bias the measurement. Several mechanisms can in principle generate such tilts and shifts.¹² Our aim in the rest of the paper is to demonstrate that the standard general equilibrium effects of a dynamic economy already do so. When hedgers dump assets on investors, investors' aggregate risk-exposure, y , changes, tilting and shifting their demand curve. We refer to the second term as the endogenous risk effect; as we will show in a dynamic asset pricing model the tilt $b^s - b^n$ is related to the change in endogenous return volatility and co-movement, akin to the amplification effect in standard heterogeneous agent models in macroeconomics and finance (e.g. Longstaff and Wang, 2012; He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014; Kondor and Vayanos, 2019). We will also explain how the shift, $a^s - a^n$, is related to intertemporal hedging amplified by general equilibrium forces.

Importantly, we will demonstrate that even if the shock to hedgers is small, implying in equilibrium only a small tilt $b^s - b^n$ or a small shift $a^s - a^n$, the second and third components still might be sizable. This is so, because the denominator is often the same order as the numerators in the last two terms of (3) in equilibrium. Intuitively, when the supply shock changes the return covariance only marginally, it also tends to change equilibrium returns at the same order of magnitude.

The tilt, shift and non-linearity: the big picture. In our model, the supply shock both tilts and shifts the demand curve. The main insight, however, goes beyond the particularities of this model. In general, a dynamic demand curve depends not only on expected returns but also on other equilibrium objects that summarize the future return distribution:

$$\hat{y}_t^\xi = D(\mathbb{E}_t(R^\xi), \Xi_t^\xi). \quad (4)$$

¹²Corell et al. (2025) and Haddad et al. (2024) suggest alternative mechanisms leading to an altered demand curve due to a supply shock.

These objects, represented by Ξ_t^ξ , also respond to the shock to residual supply. In differences and after linearization,

$$\Delta \hat{y}_t^\xi = D'_1 \cdot \Delta \mathbb{E}_t(R^\xi) + D'_2 \cdot \Delta \Xi_t^\xi. \quad (5)$$

The conceptual slope is D'_1 . It carries structural content and is therefore useful for counterfactual experiments. Instrumenting $\Delta \mathbb{E}_t(R^\xi)$ with residual supply shocks instead recovers the measured slope

$$\frac{\Delta \hat{y}_t^\xi}{\Delta \mathbb{E}_t(R^\xi)} = D'_1 + D'_2 \frac{\Delta \Xi_t^\xi}{\Delta \mathbb{E}_t(R^\xi)},$$

because the instrument is correlated with the second term in (5). The last ratio is not a property of the demand function. It measures how strongly this particular shock moves the other equilibrium objects relative to expected returns. The measured slope therefore mixes the shape of the demand curve with the equilibrium propagation of one specific shock, and loses the structural interpretation. Adding controls for Ξ_t^ξ does not alleviate the concern, as it could lead to a bad control problem (Angrist and Pischke, 2009). We return to these issues and potential remedies in Section 6.3.

3 A Model of Dynamic Asset Pricing

We build a dynamic asset pricing model with multiple assets and heterogeneous investors based on Kondor and Vayanos (2019). Our aim is to study how returns and holdings of a particular group of investors change when an exogenous shock shifts their residual supply curve. To this end, in this section we first set up the model, and then present our main objects of interest in the static benchmark.

3.1 The Dynamic Setup and Definitions

Time is continuous and infinite, indexed by $t \in [0, \infty)$.

Assets. There is a risk-free asset that yields a constant instantaneous rate of return $r > 0$, which is taken as exogenous in the model; that is, we do not clear the goods market.¹³

There are I long-lived risky assets with cumulative cash flows D_t following

$$dD_t = \bar{D}dt + \sigma^\top dB_t, \quad (6)$$

¹³By opting for an exogenous risk-free rate, in the tradition of the asset demand system literature, our analysis represents the minimal departure from Koijen and Yogo (2019); Gabaix and Koijen (2022) and therefore sharpens our economic message. One possible interpretation is that r represents the constant return on a storage technology. See Fuchs et al. (2025) (focusing on micro-elasticity in a static context) and Duarte, Gopalakrishna, Kargar, Li and Silva (2026) (focusing on macro-elasticity in a dynamic context) for a discussion of the effect of an endogenous risk-free rate in demand-elasticity estimation.

where $\bar{D}$ is a constant $I \times 1$ vector, σ is a constant and invertible $I \times I$ matrix, and $\top$ denotes transpose. Uncertainty of cash flows is described by the I -dimensional Brownian motion B_t . Each risky asset is in fixed supply, and we denote by S the $I \times 1$ vector consisting of asset supplies measured in terms of number of shares. We let $\Sigma \equiv \sigma^\top \sigma$.

Agents. The economy is populated by two types of agents: hedgers and investors. Without loss of generality, we assume that there is a unit measure continuum of each type of agents.

For simplicity, we assume that hedgers are short-lived and maximize a mean-variance objective over their instantaneous changes in wealth. Also for simplicity, we assume that each generation leaves their wealth to the next generation without consumption.¹⁴ That is, a hedger maximizes

$$\mathbb{E}_t[dv_t] - \frac{\alpha}{2} \text{Var}_t(dv_t),$$

where v_t is the hedger's time t wealth. $\alpha > 0$ is hedgers' absolute risk aversion coefficient over wealth.

On the other hand, investors are long-lived and maximize their discounted utility over intertemporal consumption. That is, an investor maximizes

$$\mathbb{E}_t \left[\int_t^\infty e^{-\rho(\tau-t)} u(\hat{c}_\tau) d\tau \right],$$

where $\hat{c}_\tau$ is the individual investor's time τ consumption, and ρ is investors' time discount rate. We assume that $\rho > r$ to ensure that investors do not accumulate infinite wealth over time. We denote by w_t investors' aggregate time- t wealth, which is one of the state variables in this economy.

Our paper focuses on Constant Relative Risk Aversion (CRRA) investors, with the following two subcases.¹⁵

Case 1 (Log investors). *Investors' instantaneous utility* $u(\hat{c}_\tau) = \log \hat{c}_\tau$.

Case 2 (General CRRA investors). *Investors' instantaneous utility* $u(\hat{c}_\tau) = \frac{\hat{c}_\tau^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$.

Hedging demand and states. Each hedger receives a random endowment $u_t^\top dD_t$ from time t to $t + dt$, where u_t is an $I \times 1$ vector and parameterizes the background risk in the economy. In our main model, u_t is modeled to follow a Markov switching process. The economy is either in a normal state or a stressed state. We use $\xi \in \{n, s\}$ to indicate the

¹⁴As we explain below, in our model short-lived hedgers' asset demand just serves as a shifter to identify the demand slope for the long-lived investors. As a result, the exact modeling of hedgers is less important for our purpose. For instance, the alternative assumption that each generation of hedgers consumes their wealth and the next generation arrives at the economy with some exogenous constant wealth would not change the analysis significantly. Kondor and Vayanos (2019) solve a version of this model with long-lived hedgers.

¹⁵In Appendix C, we also work out the case of CARA investors with $u(\hat{c}_\tau) = -e^{-\frac{\gamma}{r}\hat{c}_\tau}$. That specification gives analytical characterization for various objects of interest.

state of the economy, with n denoting the normal state and s denoting the stressed state; so $u_t \in \{u^n, u^s\}$ depending on the state of the economy. For expositional reasons we assume $u^s \geq u^n$, with u^n and u^s being $I \times 1$ vectors.

The state of the economy switches from the normal state to the stressed state with Poisson intensity $\lambda^n \geq 0$, and from the stressed state to the normal state with intensity $\lambda^s \geq 0$.

Equilibrium definition. Throughout we focus on symmetric equilibrium where economic agents are taking the same policy within each group (hedgers or investors). Denote by X_t^ξ and y_t^ξ , each $I \times 1$ vectors, the aggregate asset demand from hedgers and investors. For convenience, from this point on, we express investors' aggregate demand y_t^ξ as a share of their aggregate wealth, while X_t^ξ denotes the dollar value of hedger's aggregate demand in state ξ and time t . The endogenous price process is P_t^ξ , also with dimension of $I \times 1$. Market clearing requires that the induced asset demand from both types of agents clear the exogenous asset supply at any time, that is,

$$y_t^\xi w_t^\xi = \text{diag}(P_t^\xi)S - X_t^\xi. \quad (7)$$

It is useful to think about the right hand side as the residual supply curve from investors' point of view.

Prices and returns. We conjecture and later verify that the structure of the equilibrium price vector P_t^ξ , for $\xi \in \{s, n\}$, is

$$P_t^\xi(w_t^\xi) = \frac{\bar{D}}{r} - \pi^\xi(w_t^\xi), \quad (8)$$

where $\pi^\xi(w_t^\xi)$ is an $I \times 1$ vector describing the time-varying risk premium when investors' aggregate wealth is w_t^ξ . That is, the conjecture implies that the aggregate wealth of the investors, w_t^ξ , and the state of hedgers, ξ , are the only two state variables of the model. As the jumps in hedging demand shocks affect equilibrium prices, we conjecture that for the endogenously determined drift $\mu_{P_t}^\xi$, volatility $\sigma_{P_t}^\xi$, and jump $J_t^\xi \equiv P_t^{-\xi} - P_t^\xi$ we can write

$$dP_t^\xi = \mu_{P_t}^\xi dt + \sigma_{P_t}^{\xi\top} dB_t + J_t^\xi dN_t^\xi, \quad (9)$$

where $\{N_t^\xi : t \geq 0\}$ is a Poisson counting process with intensity λ^ξ . It will be useful to map the change in the price vector, dP_t^ξ to the (percentage) asset return vector defined as

$$dR_t^\xi \equiv \text{diag}(P_t^\xi)^{-1} (dD_t + dP_t^\xi); \quad (10)$$

throughout, we restrict the parameter space to ensure that P_t^ξ is strictly positive, and hence dR_t^ξ is well defined. Then the endogenous (percentage) asset return vector dR_t^ξ has a drift $\mu_{Rt}^\xi \equiv \text{diag}(P_t^\xi)^{-1}(\bar{D} + \mu_{Pt}^\xi)$ and volatility $\sigma_{Rt}^\xi \equiv (\sigma_{Pt}^\xi + \sigma) \text{diag}(P_t^\xi)^{-1}$ defining the return covariance matrix, $\Omega_t^\xi \equiv (\sigma_{Rt}^\xi)^\top \sigma_{Rt}^\xi$ and a jump component $J_{Rt}^\xi \equiv \text{diag}(P_t^\xi)^{-1}(P_t^{-\xi} - P_t^\xi)$. With a slight abuse of notation, we shall write

$$\mathbb{E}_t(R^\xi) \equiv \frac{\mathbb{E}_t(dR_t^\xi)}{dt} \quad \text{and} \quad \Delta \mathbb{E}_t(R) \equiv \frac{\mathbb{E}_t(dR_t^s)}{dt} - \frac{\mathbb{E}_t(dR_t^n)}{dt} \quad (11)$$

for the instantaneous expected return in state ξ and the change in instantaneous expected return when the shock hits.

Non-hedge assets. While we solve the model for general parameters, to ease the exposition, in signing these effects and discussing intuition we focus on *non-hedge assets*, for which the endogenous price volatility and the risk premium are positive, $\sigma_{Pt,i}^\xi > 0$ and $[\mathbb{E}_t(R^\xi) - r\mathbf{1}]_i > 0$.¹⁶

Hedgers' demand curve and the residual supply shock. Given the dynamics of returns, hedgers' wealth follows

$$dv_t^\xi = \left[rv_t^\xi + X_t^{\xi\top} \left(\mu_{Rt}^\xi - r\mathbf{1} \right) + \underbrace{u^{\xi\top} \bar{D}}_{\text{expected endowment dividends}} \right] dt + \left[(\sigma_{Rt}^\xi) X_t^\xi + \underbrace{\sigma u^\xi}_{\text{endowment shocks}} \right]^\top dB_t + X_t^{\xi\top} J_{Rt}^\xi dN_t^\xi, \quad (12)$$

where $\mathbf{1}$ is an $I \times 1$ vector of ones. Given the hedger's wealth and their mean-variance utility, the first-order condition implies

$$X_t^\xi = \left[\Omega_t^\xi + \lambda^\xi J_{Rt}^\xi J_{Rt}^{\xi\top} \right]^{-1} \left[\frac{\mathbb{E}_t(R^\xi) - r\mathbf{1}}{\alpha} - (\sigma_{Rt}^\xi)^\top \sigma u^\xi \right]. \quad (13)$$

This is hedgers' demand. As is apparent, when u^n switches to $u^s \geq u^n$, keeping all else fixed, hedgers' demand curve shifts downward. This translates to an upward shift in investors' residual supply curve $\text{diag}(P_t^\xi)S - X_t^\xi$, and we expect that the state s corresponds to positive flows from hedgers to investors in equilibrium.

¹⁶Both properties reflect a positive covariance of the asset's cash flows with investors' aggregate portfolio (equivalently, its price loads positively on investors' wealth, $P_i^{\xi'}(w_i^\xi) > 0$), and fail together for a hedge asset. All of our propositions hold as stated, without any additional parameter restrictions; only the signs of the endogenous-risk and price-volatility effects in the surrounding discussion depend on this restriction, which an interested reader can undo by reversing, asset by asset, the signs of $[\mathbb{E}_t(R^\xi) - r\mathbf{1}]_i$ and $P_i^{\xi'}$.

A one-off residual supply shock. While we solve the model for the general case with arbitrary λ^s and λ^n , sometimes it is convenient to focus on the case of $\lambda^s = \lambda > 0$ and $\lambda^n \rightarrow 0$. Intuitively, this corresponds to an unanticipated (MIT) shock pushing the system to the stressed state, which is expected to revert to the normal state with intensity λ , with no further shifts expected. We refer to this special case as a *one-off shock* with persistence $\frac{1}{\lambda}$.

The measured slope. There is a long tradition in economics to recover the slope of a demand curve of some agents by observing how quantities and prices are affected by shocks to the (residual) supply. We define a corresponding measure in our economy as follows.

Definition 1. *Consider a dynamic equilibrium with a one-off shock with persistence $\frac{1}{\lambda}$. The measured slope is defined as*

$$\mathcal{M}(w_t^n; \lambda, u^n, u^s) \equiv \frac{[y_t^s - y_t^n]_i}{[\Delta \mathbb{E}_t(R)]_i}. \quad (14)$$

Note that, to ease the exposition, we define measured slope with respect to a one-off shock. Therefore, $\mathcal{M}(\cdot)$ measures how an unanticipated residual supply shock with persistence $\frac{1}{\lambda}$ changes investors' demand for asset i for each unit of change in its expected return. Considering a λ^n bounded away from zero would have the expected effect on the quantitative results, without affecting the qualitative picture.

Note also that $\mathcal{M}(\cdot)$ is closer to a time-series estimator which compares the differences in demand and expected return of the same asset just before and after the shock. Having said that, in Section 4.3, we expand on why the insights presented in this paper are similarly relevant when the econometrician aims to identify the slope of investors' demand using cross-sectional variation.

Road map. Our paper aims to understand whether the measured slope recovers the slope of investor's demand curve. Conceptually, we think of the latter as the representative investor's optimal quantity response per unit change in expected returns. The analysis in the most general case of our model requires a general formal definition of the slope of investor's demand which we will provide in Section 5.2. Before that, we build up the intuition of why and when it is mismeasured by the measured slope through a series of examples and special cases.

3.2 The Static Benchmark

Just as in the dynamic model, in this static benchmark we have I assets and two groups of agents. For simplicity, we keep hedgers' demand general. We assume that for a price

vector, P , and a vector of demand shocks u , the dollar value of hedgers' aggregate demand is $X = X(P, u)$ with $X(P, u) : \mathbb{R}^I \times \mathbb{R}^I \rightarrow \mathbb{R}^I$ being an arbitrary function.

Following Campbell and Viceira (2002), we assume that the risky assets pay log-normally distributed dividends $\log D \sim N(\bar{D}, \Sigma)$ where S , $\bar{D}$ and D are $I \times 1$ while Σ is $I \times I$. Then, following Campbell and Viceira (2002) and Kojien and Yogo (2019) in relying on the log-linearized approximation, we write investors' demand curve in terms of portfolio shares as

$$\hat{y} = \frac{1}{\gamma} \Sigma^{-1} \left(\mathbb{E}(R) - r\mathbf{1} + \frac{1}{2} \text{diag}(\Sigma) \right) \quad (15)$$

where $\mathbb{E}(R) = \bar{D} - \log P$ is the expected log gross return vector. Comparing (15) to (1) maps b to the matrix $\frac{1}{\gamma} \Sigma^{-1}$ and a to $\frac{1}{2\gamma} \Sigma^{-1} \text{diag}(\Sigma)$. Importantly, in this static benchmark, both depend on deep parameters only, hence they are not sensitive to the state of the economy.

Static demand slope $\mathcal{C}^{\text{stat}}$. For didactic purposes, we map the vector-to-vector slope $\frac{1}{\gamma} \Sigma^{-1}$, into an asset specific slope as follows. Suppose an investor is asked how much she would increase her demand for asset i if the vector $\mathbb{E}(R)$ would change to $\mathbb{E}(R) + \delta$ where δ is an arbitrary vector. The static slope will measure the response by comparing the change in the investor's portfolio share to the change in the implied return, that is

$$\mathcal{C}^{\text{stat}}(\delta) \equiv \frac{[\hat{y}^h - \hat{y}]_i}{[\delta]_i} = \frac{1}{\gamma} [\Sigma^{-1}]_{ii} + \frac{1}{\gamma} \frac{\sum_{j \neq i} \Sigma_{ij}^{-1} [\delta]_j}{[\delta]_i}, \quad (16)$$

where $\hat{y}^h$ is the investor's portfolio share under the hypothetical scenario, just like what we defined in (2).

The first element in (16) is the own-slope characterizing the investor's response to a change in the return of the same asset only. The second element is due to the cross-sensitivities $\frac{1}{\gamma} \Sigma_{ij}^{-1}$ characterizing the response due to changes in the returns of other assets. If only δ_i were different from 0, or if cash-flows were uncorrelated, or if there were a single asset, the inverse of return variance (scaled by risk-tolerance) would describe the slope of the demand curve. However, in general, for a complete characterization of the static slope of any investor's demand curve, we require knowledge of the full slope matrix of $\frac{1}{\gamma} \Sigma^{-1}$.

We highlight one important takeaway from the static benchmark demand slope in (16), which will be crucial in understanding our result in a dynamic setting. While $\mathcal{C}^{\text{stat}}(\delta)$ depends on the direction of the change in returns δ , the demand curve as a function of δ is independent of the demand from the other agents (i.e., hedgers).

Note that we define the slope as the quantity response to a unit change of expected return, not to a unit change of price. In a static setting, these two definitions would coincide, because with exogenous liquidating dividends a unit drop of today's price translates one to one to a

unit increase of expected return by holding one share of asset today. However, as we will see, in a dynamic setting, the persistence of shocks and future risk matters, hence this translation no longer holds. Because researchers in financial economics think of demand for an asset as a function of its return characteristics, we believe this distinction is conceptually important.

Also, in contrast with the empirical literature, we do not normalize our measure to transform the slope to an elasticity. As HHHKL explain, this is so because *natural units* differ across modeling approaches. For instance, in the logit framework of Kojien and Yogo (2019), a percentage change in demand as a function of a percentage change in return gives the most canonical form for the demand curve, while in our CRRA framework the portfolio share as a function of percentage (or log) returns does. In fact, since relative risk aversion can be defined in terms of the investor's value function, the portfolio share indeed serves as the natural unit of analysis within the general framework of the classic Merton (1971).

Static measured slope $\mathcal{M}^{\text{stat}}$. Now we can compare $\mathcal{C}^{\text{stat}}(\delta)$ to the measured static slope $\mathcal{M}^{\text{stat}}(\delta)$, which is the equilibrium effect of a shift in residual supply on the position of investors. Shocks to hedgers' demand change the equilibrium through the market clearing condition

$$y^\xi + X(P^\xi, u^\xi) = \text{diag}(P^\xi)S,$$

for $\xi = s, n$ where investors' aggregate wealth is normalized to 1. Let $\Delta\mathbb{E}_t(R) \equiv (\bar{D} - \log P^s) - (\bar{D} - \log P^n)$ be the change in expected (log) return due to the shock.

As a shift in u leaves $\frac{1}{\gamma}\Sigma^{-1}$ unchanged, using (15), we have

$$\mathcal{M}^{\text{stat}}(u^n, u^s) \equiv \frac{[y^s - y^n]_i}{\Delta\mathbb{E}(R)_i} = \frac{1}{\gamma} \left([\Sigma^{-1}]_{ii} + \frac{\sum_{j \neq i} \Sigma_{ij}^{-1} [\Delta\mathbb{E}(R)]_j}{[\Delta\mathbb{E}(R)]_i} \right). \quad (17)$$

Comparing (16) and (17) shows that the measured slope is identical to the static slope with respect to a carefully chosen δ .¹⁷ Indeed, one can easily show that

$$\mathcal{M}^{\text{stat}}(u^n, u^s) = \mathcal{C}^{\text{stat}}(\delta)|_{\delta=\Delta\mathbb{E}(R)}. \quad (18)$$

Therefore a shift in residual supply recovers the slope of static demand.

The main insight of the paper is that this principle does not hold in a general dynamic

¹⁷Fuchs et al. (2025) refers to the second element in the bracket of expression (17) as the spill-over. One of the main insights from their paper is that in general $\mathcal{C}^{\text{stat}}(\delta) \neq \mathcal{M}^{\text{stat}}$ unless $\delta = k \cdot \mathbf{1}_i$, which is proportional to i^{th} element of unit vector (k is a scalar). That is, measured elasticity is not own price elasticity because supply shocks affect all expected returns. Our requirement of $\delta = \Delta\mathbb{E}(R^s)$ in (18) represents the implicit assumption that the econometrician understands this point, hence all the effects we discuss in the rest of the paper are over and above those originating from this spillover.

framework.¹⁸

4 Log Investors: The Anatomy of the Tilt

To introduce our effects in steps, we start the analysis of our dynamic model in some special cases. We consider log investors, a one-off shock in residual supply, and two extreme cases: when the stressed state is fully persistent, $\lambda \rightarrow 0$, and when it is fully transitory, $\lambda \rightarrow \infty$. In the former, after the shift in hedgers' demand, investors expect to remain in the stressed state indefinitely. In the latter, investors expect the stressed state to last only for an instant and return to the normal state almost immediately. These cases not only provide simplified intuition for the tilt but also allow for some analytical results.

4.1 Equilibrium with Log Investors

Before we study the limiting economy with one-off shock, we derive the equilibrium under general $\{\lambda^n, \lambda^s\}$, following standard arguments. In the main text we provide a sketch only while the details are available in Appendix A.1.3.

As in any general equilibrium model with infinitesimal agents, we need to distinguish between an investor's individual wealth $\hat{w}_t$ and investors' aggregate wealth w_t . Each individual controls $\hat{w}_t$, which coincides with the investor sector's aggregate wealth w_t only in equilibrium; in contrast, w_t serves as a state variable in this model, which each individual takes as given. The conjectured value function, which depends on both $\hat{w}_t^\xi$ and w_t^ξ , is

$$V^\xi(\hat{w}_t^\xi, w_t^\xi) = \frac{1}{\rho} \log \hat{w}_t^\xi + \bar{q}^\xi(w_t^\xi). \quad (19)$$

As is conventional with CRRA (power) utility, we define an investor's demand in terms of her chosen portfolio share, $\hat{y}_t^\xi$. Let y_t^ξ be the aggregate counterpart of $\hat{y}_t^\xi$, and the corresponding consumption variables are $\hat{c}_t^\xi$ and c_t^ξ , respectively. Then each investor's wealth dynamics is

$$\begin{aligned} d\hat{w}_t^\xi &= r\hat{w}_t^\xi dt + \hat{w}_t^\xi \hat{y}_t^{\xi\top} \left(dR_t^\xi - r\mathbf{1}dt \right) - \hat{c}_t^\xi dt \\ &= \left[(1 - \hat{y}_t^{\xi\top} \mathbf{1}) r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{w}_t^\xi \hat{y}_t^{\xi\top} \mu_{Rt}^\xi \right] dt + \hat{w}_t^\xi \hat{y}_t^{\xi\top} (\sigma_{Rt}^\xi)^\top dB_t + \hat{w}_t^\xi \hat{y}_t^{\xi\top} J_{Rt}^\xi dN_t^\xi. \end{aligned} \quad (20)$$

From (20), the total wealth dynamic for investors aggregates to

$$dw_t^\xi = \left[(1 - y_t^{\xi\top} \mathbf{1}) r w_t^\xi - c_t^\xi + w_t^\xi y_t^{\xi\top} \mu_{Rt}^\xi \right] dt + w_t^\xi y_t^{\xi\top} (\sigma_{Rt}^\xi)^\top dB_t + w_t^\xi y_t^{\xi\top} J_{Rt}^\xi dN_t^\xi, \quad (21)$$

¹⁸Most empirical studies do not attempt to identify each element of the slope matrix (which is determined as $\frac{1}{\gamma}\Sigma^{-1}$ in our example) ; instead, implicit or explicit assumptions on the structure of this matrix identify the combinations of interest, as HHHKL explain. We abstract from these dimensionality issues and focus on the higher-level question: what if the matrix itself is not invariant to supply shocks?

which gives the drift and diffusion of aggregate wealth, $\mu_t^{w,\xi}$ and $\sigma_t^{w,\xi}$. Then, substituting the conjectured value function into the representative investor's HJB equation and deriving the first-order conditions give the familiar optimal consumption rule for a log agent

$$\hat{c}_t^\xi = \rho \hat{w}_t^\xi, \quad (22)$$

and the demand function for risky assets. The next proposition spells out the latter for the following two cases of residual supply shock: i) one-off and fully persistent with $\lambda \rightarrow 0$; and ii) one-off and fully transitory with $\lambda \rightarrow \infty$.¹⁹

Proposition 1. *Suppose that investors have log utility and there is a one-off residual supply shock. Then in both limiting cases $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$, the optimal portfolio policy of log investors has the form of*

$$\hat{y}_t^\xi = (\Omega_t^\xi)^{-1} (\mathbb{E}_t(R^\xi) - r\mathbf{1}). \quad (23)$$

Note the simple form of investors' demand function (23). Similarly as we noted for our static benchmark, the slope of investors' demand is captured by the inverse of the covariance matrix of returns. Indeed, analogously to the static case, we can define the slope of demand of the log investors as follows. Suppose in state n , the investor is asked how much she would increase her demand for asset i if the vector $\mathbb{E}_t(R)$ would change to $\mathbb{E}_t(R) + \delta$ where δ is an arbitrary vector. The log slope will measure the response by comparing the change in the investor's portfolio share to the change in the implied return, that is

$$\mathcal{C}^{\log}(w_t^n; \delta) \equiv \frac{[\hat{y}_t^h - \hat{y}_t^n]_i}{[\delta]_i} = [(\Omega_t^n)^{-1}]_{ii} + \frac{\sum_{j \neq i} [(\Omega_t^n)^{-1}]_{ij} [\delta]_j}{[\delta]_i}, \quad (24)$$

where $\hat{y}_t^h$ is the investor's portfolio share under the hypothetical scenario with a higher expected return $\mathbb{E}_t(R) + \delta$. An inconsequential difference between (16) and (24) is that log investors have a constant relative risk-aversion of 1, hence the γ parameter can be omitted.

In contrast, the consequential difference is that the exogenous Σ , which is the covariance matrix of the logarithm of date-1 dividends as in the setting of Campbell and Viceira (2002), is replaced by the endogenous Ω_t^ξ , the covariance of equilibrium asset returns. To see the

¹⁹The case of $\lambda \rightarrow 0$ is easy to understand since in this limit it is as if the economy never reverts back to the normal state from the stressed state and vice-versa. The case of $\lambda \rightarrow \infty$ is a bit more subtle, as it involves the equilibrium property of the price path; for some intuition, see footnote 26.

implication, let us write the measured slope as

$$\begin{aligned}\mathcal{M}^{\log}(w_t^n; u^s, u^n) &\equiv \frac{[y_t^s - y_t^n]_i}{[\Delta \mathbb{E}_t(R)]_i} \\ &= \mathcal{C}^{\log}(w_t^n; \delta)|_{\delta=\Delta \mathbb{E}_t(R)} + \frac{[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}] (\mathbb{E}_t(R^s) - r\mathbf{1})_i}{[\Delta \mathbb{E}_t(R)]_i},\end{aligned}\tag{25}$$

where we have used (23).²⁰ Unlike in the static case, in our dynamic equilibrium with log investors (either perfectly persistent shocks $\lambda \rightarrow 0$ or transitory shocks $\lambda \rightarrow \infty$), shifts in residual supply do not recover the slope of investors' demand, $\mathcal{C}^{\log}$. The difference stems from the fact that when hedgers dump assets on investors, the persistent change in the allocation of risk changes the covariance matrix of returns. Hence, the change in residual supply tilts investors' demand.²¹ Just as in equation (3), the resulting second term in (25) is the endogenous risk effect.

To build more intuition on the sources and direction of the tilt, we consider two examples. First, we consider the case of a single risky asset. In the second example, we consider multiple assets out of which the shift in residual supply affects only two, but in exactly opposite ways. As it turns out, we obtain sharp analytical characterizations of the tilt in two scenarios: in the first case, as $\lambda \rightarrow \infty$; and in the second case, as $\lambda \rightarrow 0$.

4.2 Single Risky Asset

With a single risky asset and log investors, the slope of demand $\mathcal{C}^{\log}$ simplifies to the inverse of return variance, $(\sigma_{Rt}^n)^{-2}$. For $\mathcal{M}^{\log}$, when hedgers demand less of the risky asset in state s for given expected return, investors hold a riskier portfolio. Therefore, we expect the volatility of their wealth to increase, which, through (8), is expected to increase the volatility of the risky asset. This should imply that the measured slope, $\mathcal{M}^{\log}$, is smaller than $\mathcal{C}^{\log}$.

The following proposition gives the formal statement about the negative endogenous risk effect. It is tempting to conjecture that when the hedging shock is extremely short-lived ($\lambda \rightarrow \infty$), this effect vanishes. We prove that this is never the case: $\mathcal{M}^{\log}$ is strictly smaller than $\mathcal{C}^{\log}$ in this limit, even when the demand shift is infinitesimally small.

Proposition 2. *Consider an economy with log investors, single risky asset $I = 1$, and the demand shift shock is transitory $\lambda \rightarrow \infty$. In this economy,*

1. *for any given $u^s > u^n$, the variance of the asset increases, $\sigma_{Rt}^n < \sigma_{Rt}^s$, hence the*

²⁰In general, we find that the second term is well defined and bounded away from zero in all relevant limits. We show this formally for various cases in Proposition 2 and Proposition 3.

²¹Note that our analysis on log investors provides a sharp comparison to Koijen and Yogo (2019) who also consider (myopic) log investors. The difference stems from the fact that their Assumption 1 assumes away the possibility that the residual supply shock affects the volatility of assets. In contrast, we show that general equilibrium forces imply that this is likely to be the case.

endogenous risk effect is negative.²² That is,

$$\mathcal{M}^{log}(w_t^n; u^n, u^s) - \mathcal{C}^{log}(w_t^n; \delta)|_{\delta=\Delta\mathbb{E}_t(R)} = \frac{\left[\frac{1}{(\sigma_{Rt}^s)^2} - \frac{1}{(\sigma_{Rt}^n)^2}\right](\mathbb{E}_t(R^s) - r)}{\Delta\mathbb{E}_t(R)} < 0. \quad (26)$$

2. Even if the demand shift shock is diminishingly small, (26) is bounded away from zero:

$$\lim_{u^n \rightarrow u^s} \left(\mathcal{M}^{log}(w_t^n; u^n, u^s) - \mathcal{C}^{log}(w_t^n; \delta)|_{\delta=\Delta\mathbb{E}_t(R)} \right) < 0.$$

The first part of Proposition 2 states that in our economy, as hedgers liquidate some of their positions due to the shock, the endogenous return volatility goes up. Hence, the endogenous risk effect is negative and the measured slope is flatter than its conceptual counterpart. Intuitively, if investors hold more of the risky asset in equilibrium due to hedgers' shock, $y_t^s > y_t^n$, it implies that a future adverse shock, $dD_\tau < 0$ results in a larger drop in investors' aggregate wealth w_τ , increasing risk premium more; and this leads to a larger decrease in price, which amplifies the initial drop in aggregate wealth. As this amplification effect is stronger when $y_t^s > y_t^n$, the shock implies a volatility hike: $\sigma_{Rt}^n < \sigma_{Rt}^s$. Since in the single asset the risk premium is positive $\mathbb{E}_t(R^s) - r > 0$, (26) holds.

We highlight that this is a form of well-established “leverage effect”—the empirical regularity that volatility rises when stock prices fall (Black, 1976; Christie, 1982), and the economic intuition is closely related to the amplification effect in various heterogeneous agent models (e.g. Longstaff and Wang, 2012; He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014; Kondor and Vayanos, 2019).²³ Importantly, the leverage effect we isolate operates *without* any financial leverage: volatility rises not because a price decline mechanically raises a firm's debt-to-equity ratio, but because investors endogenously bear more risk in equilibrium. This aligns with recent evidence that the leverage effect appears even in firms with no debt (Hasanhodzic and Lo, 2019) and in the volatility of unlevered assets (Choi and Richardson, 2016), pointing to a risk-based rather than balance-sheet origin, in the volatility-feedback tradition of Campbell and Hentschel (1992) and Bekaert and Wu (2000); indeed, recent estimates attribute a large share of index volatility variation to past returns (Guyon and Lekeufack, 2023). Our contribution is to give this risk-based leverage effect a general-equilibrium micro-foundation and to quantify its consequences for demand estimation. The

²²Note that if instead we assumed $u^s < u^n$, then $\sigma_{Rt}^n > \sigma_{Rt}^s$ and $\Delta\mathbb{E}_t(R) < 0$ implying that by (26) the endogenous risk effect would be still negative.

²³In He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014) intermediaries play the role of investors. In He and Krishnamurthy (2013) the amplification is due to the endogenous equity constraint induced by moral hazard (of intermediaries). To see the parallel between Brunnermeier and Sannikov (2014) and our setup, consider the $\gamma \rightarrow 0$ limit in our case. Then, in both models intermediaries/investors are the efficient holders of risk. However, foreseeing that they cannot operate when their wealth shrinks to zero, intermediaries/investors are willing to hold less and less risk as their capital shrinks. (For a larger γ this effect is stronger by risk-aversion.)

logic parallels a known measurement gap in this literature: just as the *measured* return–volatility correlation understates its true value at high frequency (Aït-Sahalia, Fan and Li, 2013), here the *measured* demand slope understates its conceptual counterpart.

Note that our economy features incomplete markets: investors can trade Brownian risk but cannot trade jump risk ex ante. This is consistent with our focus on one-off shocks, which arrive unanticipated ($\lambda^n \rightarrow 0$): in the normal state agents assign zero intensity to the transition, so neither would establish a jump-risk position even if such assets were traded. The key qualitative takeaway is nonetheless robust to making anticipated jump risk tradable. Take the negative endogenous risk effect: its driver is that investors change their holdings of the shocked asset after the distress shock, $y_t^s > y_t^n$. But this position change merely reflects risk sharing between hedgers and investors, and a complete market implements optimal risk sharing rather than eliminating risk. Investors therefore still absorb the asset once the economy enters the stressed state, and the core economics goes through.

A formal argument of this negative endogenous risk effect is useful. From the definition of return volatility (9)–(10) we have $\sigma_{Rt}^\xi = (\sigma + \sigma_{Pt}^\xi)/P_t^\xi$, where the first (second) term captures the exogenous (endogenous) volatility of Brownian dividends (asset price). The amplification mentioned above applies to the second component regarding the endogenous price volatility σ_{Pt}^ξ of the risky asset, which is determined in equilibrium by (using Itô’s lemma on (8) and matching volatility terms):

$$\underbrace{\sigma_{Pt}^\xi}_{\text{price volatility}} = \underbrace{P^{\xi'}(w_t)}_{>0, \text{ price loading on wealth}} \times \underbrace{\overbrace{\text{Investors' Holding}_t}^{\text{increasing in } y_t^\xi} \cdot (\sigma + \sigma_{Pt}^\xi)}_{\text{wealth } w_t \text{'s exposure to risky asset}}. \quad (27)$$

That is, the price volatility of the risky asset increases when investors increase their holdings.

When $\lambda \rightarrow \infty$, risky asset prices and aggregate wealth converge across the two states.²⁴ The convergence, however, does not imply that the relevant return moments converge. To accommodate a given demand shift shock $u^s > u^n$, investors’ endogenous risky asset holdings must differ across the two states. As a result, the stressed state features both a strictly higher return volatility and a strictly higher expected return, due to investors’ greater risk exposure in the state.

In the second part of Proposition 2, we additionally let the demand shift shock become diminishingly small, $u^s \rightarrow u^n$. In this limiting case, both the volatility difference and the expected-return difference vanish. However, the endogenous risk effect is governed by the ratio of these two first-order changes. In the proof of the proposition, we show that in the limit

²⁴We show in the Appendix that when $\lambda \rightarrow \infty$, we have $P^s(\cdot) \rightarrow P^n(\cdot)$ and $w^n \rightarrow w^s$. Intuitively, price paths should be continuous under immediate (and almost certain reversal), which differs from van Binsbergen et al. (2025) who analyze discontinuous price paths.

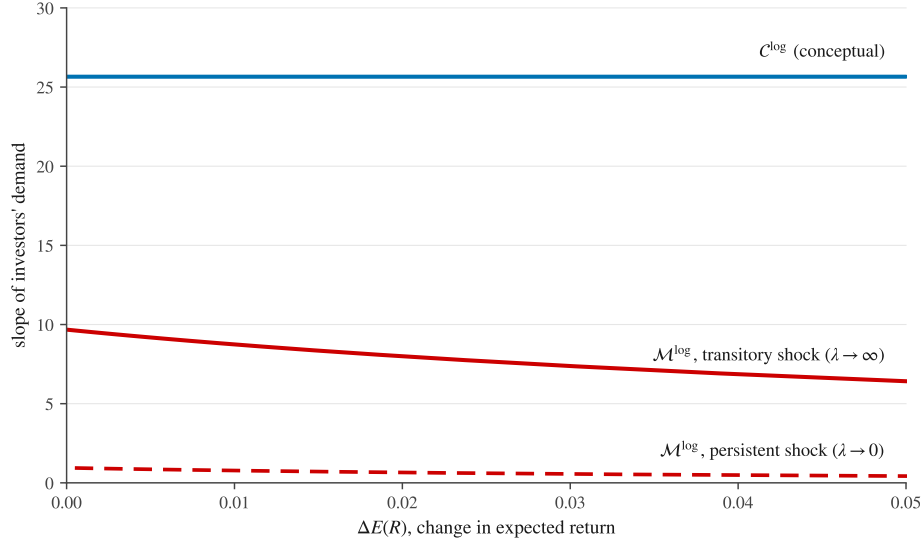


Figure 1: Measured slope of a log investor, $\mathcal{M}^{\log}$, and the corresponding conceptual slope, $\mathcal{C}^{\log}(\delta)|_{\delta=\Delta\mathbb{E}_t(R)}$ with one risky asset. The solid red line plots $\mathcal{M}^{\log}$ for a transitory shock ($\lambda \rightarrow \infty$) and the dashed red line for a persistent shock ($\lambda \rightarrow 0$); the blue line is the conceptual slope $\mathcal{C}^{\log}$, which is common to the two cases. We change the u^s shock to get varying incremental expected return (in percentage) $\Delta\mathbb{E}_t(R)$. We plot our measures against the implied change in percentage incremental expected return, with our benchmark parameterization: $\bar{D} = 0.415$, $\sigma = 0.57$, $\alpha = 1$, $r = 0.03$, $\rho = 0.08$, $S = 1$, $u^n = 0$; all curves are evaluated at the mean of the stationary distribution of investors' wealth, $\bar{w} = 3.24$. The choice of parameter values is discussed in Appendix E.3.

this ratio is well defined and bounded away from zero corresponding to the local derivative $\partial(\sigma_{Rt})^{-2}/\partial\mathbb{E}_t(R)$ along the equilibrium response to the residual supply shock. Hence, the endogenous risk effect is strictly negative even in the joint limit of a fully transitory and diminishingly small demand shift shock.

We finish this section with an illustration of the difference due to this effect in the fully persistent ($\lambda = 0$) and fully transitory cases ($\lambda = \infty$). Placing the implied change of expected return $\Delta\mathbb{E}_t(R)$ on the x -axis, Figure 1 shows $\mathcal{M}^{\log}$ and $\mathcal{C}^{\log}(\delta)|_{\delta=\Delta\mathbb{E}_t(R)}$ for various values of u^s for these two cases, with parameters calibrated to Gabaix and Koijen (2022) who study the macro elasticity.²⁵ While the difference between $\mathcal{M}^{\log}$ and $\mathcal{C}^{\log}$ is significantly lower when the shock is transitory, in line with Proposition 2, the measured slope is too flat and the mismeasurement is bounded away from zero even for infinitesimal shocks: the figure shows that $\mathcal{M}^{\log}$ remains below half of $\mathcal{C}^{\log}$ even as $\Delta\mathbb{E}_t(R) \rightarrow 0$.

²⁵The details of the calibration and the baseline parameter values are discussed in Appendix E.3. As we explain there, $\lambda \approx 1$ would match the persistence of flows in Gabaix and Koijen (2022).

4.3 Multiple Assets: The Economy with Blue and Red Assets

Next, we consider a stylized multi-asset example to illustrate how moving from return variance to the covariance matrix affects our intuition.

We picture a market with a large number of assets with the same cash-flow risk separated into two groups: red and blue. The only difference between these assets is in the corresponding component of hedgers' demand shifter, u^ξ ; in particular, hedgers hold blue assets in larger proportion due to the difference in their background risk. We imagine that when the state switches from n to s , the first and last assets, $i = 1$ and $i = I$, switch color. The first asset turns from blue to red, while the last asset turns from red to blue. The shock does not affect any other asset. That is, in state s hedgers reduce their position in the first asset but increase their position in the last asset. A simple interpretation is that blue assets are part of an index in which the last asset replaces the first asset. However, other interpretations along hedgers' mandate for blue assets (akin to Kojien and Yogo (2019)) would work equally well.

For a precise formulation, suppose that $\Sigma = \sigma^2 \mathbf{I}$, $S = s\mathbf{1}$, $\bar{D} = \bar{d}\mathbf{1}$ where $\mathbf{I}$ is the identity matrix, and s , σ^2 , and $\bar{d}$ are arbitrary scalars. The demand shifter for hedgers is as

$$[u^n]_i = \begin{cases} v & \text{for } i \leq I_1, \\ v + \Delta v & \text{for } i > I_1, \end{cases}$$

and

$$[u^s]_i = \begin{cases} v + \Delta v & \text{for } i = 1, \\ v & \text{for } i = I, \\ [u^n]_i & \text{otherwise,} \end{cases}$$

with v and Δv being some positive constants. Here, the first I_1 assets are initially blue.

We constructed this example to minimize the effect of the state ξ on the aggregate economy. After all, only the labels of two assets with the same cash-flow risk are swapped. Note, however, the aggregate wealth of investors still changes due to the shock. As each investor holds more of asset I than of asset 1 in state n , when the prices of these assets jump due to the shock, investors' wealth jumps as well. As w_t^ξ is a state variable, this affects the aggregate economy.

To sharpen our theoretical results, therefore, we consider the limit when the difference in hedgers' exposure to blue and red assets, and consequently, the change in their demand captured by Δv , is diminishingly small. As the next lemma states, in this limit the effect of the shock on the aggregate state, w_t^ξ is second order.

Lemma 1. *Consider the limit $\lambda \rightarrow 0$ in the economy where red and blue assets have the same cash-flow risk. In this economy, $\lim_{\Delta v \rightarrow 0} \frac{w_t^\xi - w_t^n}{\Delta v} = 0$.*

One might expect that the effect of this small local shock on the equilibrium outcome is negligible, implying that the endogenous risk effect disappears and, therefore, $\mathcal{M}^{\log}(\cdot)$ corresponds to $\mathcal{C}^{\log}$. We show in the next statement that this is not the case.

Proposition 3. *Consider the limit $\lambda \rightarrow 0$ in the economy where red and blue assets have the same cash-flow risk. In this economy, for sufficiently small Δv ,*

1. *the endogenous risk effect is negative and is strictly bounded away from zero. Hence*

$$\lim_{\Delta v \rightarrow 0} \mathcal{M}^{\log}(w_t^n; u^n, u^s) < \lim_{\Delta v \rightarrow 0} \mathcal{C}^{\log}(w_t^n; \delta)|_{\delta=\Delta \mathbb{E}_t(R)}.$$

2. *In particular, the variance of asset 1 due to the shock increases,*

$$\lim_{\Delta v \rightarrow 0} \frac{[\Omega_t^s - \Omega_t^n]_{11}}{\Delta v} = 2\sigma^4 \frac{\bar{F}(w_t)(s+v)\frac{\bar{D}}{r} + \pi(w_t)}{(P_t)^3} > 0$$

just as its covariance with any other asset

$$\lim_{\Delta v \rightarrow 0} \frac{[\Omega_t^s - \Omega_t^n]_{1j}}{\Delta v} = \bar{F}(w_t) \sigma^4 \frac{\bar{D}}{r} \frac{s+v}{(P_t)^3} > 0$$

where $\bar{F}(\cdot)$ is a scalar function defined in the proof and $j \neq 1, I$. (Note that by symmetry $\lim_{\Delta v \rightarrow 0} \frac{[\Omega_t^s - \Omega_t^n]_{II}}{\Delta v} = 0$).

First note the implicit result in Proposition 3 that even if the shock is diminishingly small, $\mathcal{C}^{\log}$ and $\mathcal{M}^{\log}$ are still well defined and bounded away from zero. This is because the change in quantities and the change of expected returns diminish in the same order. Second, the diminishingly small shock does not lead to a diminishingly small difference in $\mathcal{C}^{\log} - \mathcal{M}^{\log}$ either. This is because both the denominator and the numerator of the second term in (25) are of the order Δv . That is, while small shocks lead to small changes in all equilibrium objects as expected, the mismeasurement depends on the ratio of the effect on the covariance and on expected returns.

We also state that just as in the single asset case, the measured slope is flatter than its conceptual counterpart. It is because as hedgers liquidate their holdings of the first asset, both its endogenous return volatility and its endogenous covariance with all the unaffected assets go up. The intuition is similar to the single asset case. Investors, by construction, have larger exposure to the dividend shocks of red assets. Whenever future cash-flows of red assets are lower, investors' aggregate wealth drops more, decreasing their risk-bearing capacity. While this will increase the risk premia of all assets, it will lead to a disproportionate drop in the prices of red assets as investors have to keep holding more of them. Therefore red assets are more volatile and have a larger covariance with all other assets.

Cross-section versus time-series identification. It is a good place to highlight that as the empiricists exploit various instruments to identify exogenous variation in residual supply, our main insight – that residual supply affects investors’ slope of demand – holds regardless of whether the instruments operate along the time-series or cross-sectional dimension.

As we mentioned, our definition of $\mathcal{M}(\cdot)$ in (1) is closer to the time-series notion as it focuses on the differences in demand and expected return of the same asset before and after the shock. Alternatively, in economies where two assets, i and j , are close-to-identical in their fundamentals, we could also consider the definition

$$\mathcal{M}^{\text{cross-section}}(w_t^n; u^n) \equiv \frac{[y_t^n]_i - [y_t^n]_j}{[\mathbb{E}_t(R^n)]_i - [\mathbb{E}_t(R^n)]_j}, \quad (28)$$

comparing the differences in demand and expected return between two similar assets in the same cross-section in state n . This would be closer in spirit to the empirical methodology in Kojien and Yogo (2019).

In our illustrative example with blue and red assets, it is easy to contrast the two approaches. Let $i = 1$ and $j = I$, therefore $\mathcal{M}^{\text{cross-section}}$ compares a blue asset and a red asset in the cross-section $\xi = n$. $\mathcal{M}^{\text{log}}(w_t^n; u^n, u^s)$ compares an asset which is blue before the shock and red after. By Lemma A.1, in this example each asset’s position and expected return are the same state-invariant functions of (w_t^ξ, u_i) , so $\mathcal{M}^{\text{cross-section}}$ and $\mathcal{M}^{\text{log}}$ compare the same change from v to $v + \Delta v$; they differ only in that $\mathcal{M}^{\text{log}}$ additionally moves aggregate wealth from w_t^n to w_t^s . The numerator and denominator of each are first order in Δv , whereas by Lemma 1 the wealth jump $w_t^s - w_t^n$ is second order; hence the wealth-jump term is a second-order correction to first-order quantities and vanishes in the limit, giving the following Proposition.

Proposition 4. *Consider the limit $\lambda \rightarrow 0$ in the economy where red and blue assets have the same cash-flow risk. Then*

$$\lim_{\Delta v \rightarrow 0} \mathcal{M}^{\text{log}}(w_t^n; u^n, u^s) = \lim_{\Delta v \rightarrow 0} \mathcal{M}^{\text{cross-section}}(w_t^n; u^n).$$

This implies that Proposition 3 holds regardless of which approach the econometrician follows, illustrating the generality of our main insights.

5 The General Case: Intertemporal Hedging and Persistence

Now we turn to the general model with CRRA utility and residual supply shocks with arbitrary persistence, $1/\lambda$. First, we summarize the main properties of the equilibrium. Then we show how the investor’s first-order condition defines a general concept of the slope of investors’ demand, the *dynamic slope*. Finally, we show that when the residual supply shock

hits, investor's demand curve not only tilts due to the endogenous risk effect, but also shifts due to additional effects stemming from general equilibrium forces connected to intertemporal hedging and the persistence of the shock.

5.1 Equilibrium in the General Case

We derive the equilibrium in our multi-asset dynamic model. As main ingredients, we already defined the conjectured price process, hedgers' demand, and investor's individual and aggregate wealth in equations (9) and (13) and (20)-(21).

Investors' demand. Given the dynamics of the individual and aggregate wealth of investors, an investor's value function in state $\xi \in \{n, s\}$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation which we write in a general form:

$$\begin{aligned} \rho V^\xi = \max_{\hat{c}_t^\xi, \hat{y}_t^\xi} : & \left\{ u(\hat{c}_t^\xi) + V_{\hat{w}}^\xi \mu_t^{\hat{w}, \xi} + \frac{1}{2} V_{\hat{w}\hat{w}}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{\hat{w}, \xi} + V_w^\xi \mu_t^{w, \xi} + \frac{1}{2} V_{ww}^\xi (\sigma_t^{w, \xi})^\top \sigma_t^{w, \xi} \right. \\ & \left. + V_{\hat{w}w}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{w, \xi} + \lambda^\xi (V^{-\xi, +} - V^\xi) \right\}. \end{aligned}$$

Here, V^ξ stands for the value function $V^\xi(\hat{w}_t^\xi, w_t^\xi)$; $V^{-\xi, +}$ is the value function evaluated after the jump, with the superscript $+$ denoting post-jump while $-\xi$ denoting the other state once jump occurs:

$$V^{-\xi, +} \equiv V^{-\xi} \left(\hat{w}_t^\xi (1 + \hat{y}_t^{\top} J_{Rt}^\xi), w_t^\xi (1 + y_t^{\top} J_{Rt}^\xi) \right).$$

Denote $\Omega_t^\xi \equiv (\sigma_{Rt}^\xi)^\top (\sigma_{Rt}^\xi)$ to be the covariance matrix of returns. The first-order condition in $\hat{y}_t^\xi$ gives the familiar components of asset demand as in Merton (1971):

$$\begin{aligned} \hat{y}_t^\xi = & \underbrace{\frac{\overbrace{V_{\hat{w}}^\xi}^{\text{risk tolerance}}}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1} \left(\mathbb{E}_t(R^\xi) - r \right)}_{\text{myopic}} + \underbrace{\frac{V_{\hat{w}w}^\xi}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} w_t^\xi y_t^\xi}_{\text{hedging of Brownian}} + \underbrace{\frac{1}{\hat{w}_t^\xi} \frac{V_{\hat{w}}^{-\xi, +} - V_{\hat{w}}^\xi}{(-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1} \lambda^\xi J_{Rt}^\xi}_{\text{hedging for reversal, } J_{Rt}}. \end{aligned} \quad (29)$$

The first part, which is myopic demand, is analogous to the static demand curve: the instantaneous expected return scaled by the product of the inverse of the covariance matrix and risk-tolerance. One could also map the term b^ξ in (1) in our example to $\frac{V_{\hat{w}}^\xi}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1}$. As we will see, the shock to residual supply affects this term through the endogenous covariance matrix, and this endogenous risk effect is responsible for the tilt of the demand curve.

The second part is the intertemporal hedging demand with respect to the diffusion part of the aggregate state variable, w_t . When w_t affects the marginal value of a dollar, its correlation with the investor's individual wealth $\hat{w}_t$ affects her demand. The last part is a similar term

but it represents hedging demand with respect to the reversal. When the residual supply curve reverts, prices jump, affecting investor's marginal value of a dollar and their individual wealth at the same time. Hedging this affects her demand.

These two hedging terms in (29) map to the term a^ξ in (1), and they may be an empirically important ingredient for understanding asset pricing (Campbell et al., 2018, 2023). As we will see, these terms are affected by the residual supply shock in a complex way, resulting in a shift of the demand curve.

Solving for equilibrium. We solve for the equilibrium prices after imposing that individual positions equal the aggregate position $\hat{y}_t^\xi = y_t^\xi$ and market clearing for each instant

$$X_t^\xi + w_t^\xi y_t^\xi = \text{diag}(P_t^\xi)S.$$

The next proposition spells out the resulting dynamic demand function and the structure of equilibrium for the cases of log investors and CRRA investors with arbitrary state switching intensities $\{\lambda^s, \lambda^n\}$.

Proposition 5. *In Case 1 (log investors), the value function has the form of*

$$V^\xi(\hat{w}_t^\xi, w_t^\xi) = \frac{1}{\rho} \log \hat{w}_t^\xi + \bar{q}^\xi(w_t^\xi). \quad (30)$$

The optimal consumption-portfolio policies are

$$\begin{aligned} \hat{c}_t^\xi &= \rho \hat{w}_t^\xi, \\ \hat{y}_t^\xi &= (\Omega_t^\xi)^{-1} \left(\mathbb{E}_t(R^\xi) - r\mathbf{1} \right) + \lambda^\xi (\Omega_t^\xi)^{-1} \left(\frac{1}{1 + \hat{y}_t^{\xi\top} J_{Rt}^\xi} - 1 \right) J_{Rt}^\xi. \end{aligned} \quad (31)$$

The ODE that determines $\bar{q}^\xi(w_t^\xi)$ is in Appendix A.1.3.

In Case 2 (CRRA investors), the value function has the form of

$$V^\xi(\hat{w}_t^\xi, w_t^\xi) = \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} q^\xi(w_t^\xi), \quad (32)$$

where $q^\xi(w_t^\xi)$ is the marginal value of wealth in state w_t^ξ . The optimal consumption-portfolio

policies are

$$\begin{aligned}\hat{c}_t^\xi &= q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} \hat{w}_t^\xi; \\ \hat{y}_t^\xi &= \frac{1}{\gamma} (\Omega_t^\xi)^{-1} \left(\mathbb{E}_t(R^\xi) - r\mathbf{1} \right) + \frac{1}{\gamma} \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi y_t^\xi + \frac{\lambda^\xi}{\gamma} (\Omega_t^\xi)^{-1} \left(\frac{q^{-\xi}(w_t^\xi(1+y_t^{\xi\top} J_{Rt}^\xi))}{q^\xi(w_t^\xi)[1+\hat{y}_t^{\xi\top} J_{Rt}^\xi]^\gamma} - 1 \right) J_{Rt}^\xi.\end{aligned}\tag{33}$$

The ODEs determining $q^\xi(w_t^\xi)$ are given in Appendix A.1.4.

Note that investors' demand in each of our special cases follows the general structure of (29). However, (31) illustrates that in the simpler log case there is no intertemporal hedging term for the Brownian shock. This is a well-known property pointed out by Merton (1971). At the same time, the intertemporal hedging term for the reversal remains.²⁶

Importantly, because of the form of the equilibrium pricing vector (8), investors' portfolio choice changes not only with the state of the residual supply curve, ξ , but also with their aggregate wealth, w . Because poorer investors are less willing to absorb risk, aggregate investors' wealth affects the risk premium.

5.2 Dynamic Slope and Myopic Slope

How should we think about the concept of slope of our investors' demand function in this general dynamic environment? In a static benchmark with a single asset, the slope, which measures the response in the asset holding to a unit change in its return, is a scalar. As (15) illustrates, with multiple assets, a matrix describes how each component of the portfolio changes as the returns of assets do.

The general dynamic environment is more complex. As the investment opportunity set can predictably fluctuate, the investor will consider all current instantaneous (expected) returns as well as the distribution of future return paths when making her portfolio decision. Conceptually, one should measure the response in $\hat{y}_t$ for any marginal change at any point of these return paths. This is an infinite-dimensional object.

The power of the Merton approach comes from summarizing the joint distribution of all future price paths into a small number of simpler economic objects. As (29) illustrates, apart from the expected return vector and covariance matrix, the Merton investor's demand also responds to changes in the future investment opportunity sets in all periods $\tau > t$. But this also suggests that our heuristic approach in defining the slope of the investors' demand curve for our special cases, that is, $\mathcal{C}^{\min}$, $\mathcal{C}^{\text{stat}}$, and/or $\mathcal{C}^{\log}$ is not sufficient. When intertemporal

²⁶Equation (31) also echoes the simple optimal myopic portfolio decision in Proposition 1. The subtlety of the $\lambda \rightarrow \infty$ case is that in Appendix A.2 we prove the associated return jump shrinks to zero, so that, while $J_{Rt} \rightarrow 0$, $\lim_{\lambda \rightarrow \infty} \lambda J_{Rt}$ remains finite and the term inside the bracket (with the order of $\hat{y}_t^{\xi\top} J_{Rt}^\xi$) converges to zero.

hedging is present, the investor's response to an increase in expected returns from $\mathbb{E}_t(R)$ to $\mathbb{E}_t(R) + \delta$ will depend on how future price paths change beyond the next instant.

Based on the discussion above, we suggest two alternatives.

Myopic slope. When simplicity is preferred, one can focus on the myopic component of demand and consider the matrix $\frac{1}{\gamma}(\Omega_t^\xi)^{-1}$ as an approximation of the conceptual slope of the demand function. Then, analogously to the static conceptual demand (16), we define the measure of myopic conceptual slope of the CRRA investor in the normal state as

$$\mathcal{C}^{\text{myo}}(w_t^n; \delta) \equiv \underbrace{\frac{1}{\gamma} [(\Omega_t^n)^{-1}]_{ii}}_{\text{own-effect}} + \underbrace{\frac{\sum_{j \neq i} (\Omega_t^n)_{ij}^{-1} [\delta]_j}{\gamma [\delta]_i}}_{\text{cross-asset spillover}}. \quad (34)$$

Dynamic slope. The second approach is to parameterize the future path of returns beyond the next instant, capturing the idea that the investor cares about the persistence of the shock to expected returns, its size and relative magnitude across assets (i.e., its direction), and also the flow share of additional return versus the lump-sum when the shock reverses (i.e., its composition). Our headline measure is *the dynamic slope* defined as follows. We then characterize it fully in our equilibrium model.

Definition 2. *[Dynamic Slope] Suppose that an investor is offered the opportunity to buy any asset i with a δ_i additional instantaneous expected return where δ is an arbitrary vector. In particular, she is asked to form demand $\hat{y}_t^h$ facing, for each asset i , the dividend process*

$$dD_{i,t}^h = \bar{D}_i dt + [\sigma^\top dB_t]_i + (1 - \chi_i)[P_t^n]_i \delta_i dt + \chi_i \frac{[P_t^n]_i \delta_i}{\lambda} dN_t^h, \quad (35)$$

instead of (6), where dN_t^h is a Poisson process with intensity λ . After the first jump the dividend process switches to (6) forever. Then the dynamic slope is measured as

$$\mathcal{C}^{\text{dyn}}(w_t^n; \theta) \equiv \frac{[\hat{y}_t^h - \hat{y}_t^n]_i}{\delta_i}. \quad (36)$$

where $\hat{y}_t^n$ refers to the equilibrium demand in a normal state where $\lambda^n = 0$, and the tuple $\theta \equiv \{\lambda, \delta, \chi\}$ parameterizes the persistence, direction and composition of the shock to return path, respectively. Here χ is an $I \times 1$ vector with $\chi_i \in [0, 1]$, so that each asset i has its own split between the flow and the lump-sum component of the higher return.

The concept of dynamic slope is appealing for several reasons. First, it is simple to understand. It measures the optimal response of an investor's demand to a change in expected returns (implemented through a hypothetical dividend process) of given persistence, direction,

and composition. Note that under our parameterization of (36), the change in expected return is always δ irrespective of λ and χ .

Second, as we argue below, it allows for a change in return path sufficiently close to its equilibrium counterpart in our dynamic setting. In our model, a shift in residual supply governed by a Poisson process implies, apart from an increase in instantaneous expected returns, a higher flow return and an added lump-sum at reversal. The parameter χ_i governs, for each asset i , the share of the higher return which is received as a flow relative to the share received as a lump sum. The inverse of parameter λ governs the persistence of the improvement of the investment opportunity set. In the $\lambda \rightarrow \infty$ limit, dynamic slope captures the investor's response to a hike (or drop) in expected return for an instant. In the other extreme, the $\lambda \rightarrow 0$ case captures a permanent change in expected return.

Third, dynamic slope is closely tied to typical policy questions addressed in the growing literature of demand elasticity. For instance, the central question of quantitative easing (QE) is: to maintain the yield of certain bonds at z basis points below their non-intervention equilibrium level, how much must the Federal Reserve purchase as a fraction of outstanding bonds over a specific period? Dynamic slope provides the critical individual demand characteristic needed to answer this question: the change in the amount an investor is willing to hold in response to a change in return of given persistence.

Finally, observe that just as our definition for the static case, dynamic slope captures the quantity response to a given change in the expected return vector, not to a given change in the price vector. In the dynamic context, the distinction is critical. Clearly, as it has been pointed out by Davis et al. (2023) and van Binsbergen et al. (2025), the same price shock implies a much lower expected return shock if it is persistent than if it is transitory. Therefore, throughout our analysis we control for this by keeping the expected return shock constant in our comparative statics.

Dynamic slope as an optimization problem. To calculate dynamic slope, we first solve for the equilibrium as described in Proposition 5 with a constant residual supply curve, that is, $u = u^n$ and $\lambda^n = 0$. This defines the portfolio holdings $\hat{y}^n$ and value function $V^\xi(\hat{w}_t^\xi, w_t^\xi)$ when the economy is under the baseline normal state. Also, the equilibrium objects μ_t^w, σ_t^w represent the dynamics of the aggregate state variable which is not affected by the hypothetical dividend distribution (because the extra dividends are offered only to a given investor).

Then, we introduce $V^h(\cdot, \cdot)$, the value function of that investor who is receiving the hypothetical dividends (35). We conjecture that in the CRRA case, it has the form of

$$V^h(\hat{w}_t^h, w_t^n) = q^h(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma}.$$

Note that the marginal value of a dollar $q^h(w_t^n; \theta)$ depends on the parameters of the dividend process. Also, the HJB equation of this problem takes into account that when the offer ends so the dividend process reverts back to its equilibrium dynamics, this investor's value function jumps back to that under normal times.

Given these objects, we solve for the optimal portfolio choice $\hat{y}^h$ of the investor facing hypothetically higher dividends. Comparing to the investor's demand in normal times $\hat{y}^n$ and substituting both into the definition of dynamic slope, (36) gives a similar decomposition of dynamic slope as in the dynamic asset demand in (29). The following proposition summarizes the results for both the log and the CRRA cases, with details provided in Appendix A.5.

Proposition 6. *In Case 1 (log investors), demand under hypothetical dividend path (35) is*

$$\hat{y}_t^h = (\Omega_t^n)^{-1} (\mu_{Rt}^n - r\mathbf{1} + \delta) + \left[\frac{\lambda}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi)\delta} - 1 \right] (\Omega_t^n)^{-1} \text{diag}(\chi)\delta. \quad (37)$$

Hence, the dynamic slope has only two components:

$$\begin{aligned} \mathcal{C}^{dyn}(w_t^n; \theta) &= \frac{[\hat{y}_t^h - \hat{y}_t^n]_i}{\delta_i} \\ &= \underbrace{[(\Omega_t^n)^{-1}]_{ii} + \frac{\sum_{j \neq i} [(\Omega_t^n)^{-1}]_{ij} [\delta]_j}{[\delta]_i}}_{\text{myopic slope}} + \underbrace{\left[\frac{\lambda}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi)\delta} - 1 \right] \frac{[(\Omega_t^n)^{-1} \text{diag}(\chi)\delta]_i}{[\delta]_i}}_{\text{hedging against reversal}}. \end{aligned} \quad (38)$$

In Case 2 (CRRA investors) demand under hypothetical dividend path (35) is

$$\begin{aligned} \hat{y}_t^h &= \frac{1}{\gamma} (\Omega_t^n)^{-1} (\mu_{Rt}^n - r\mathbf{1} + \delta) + \frac{1}{\gamma} \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} w_t^n y_t^n \\ &\quad + \frac{1}{\gamma} (\Omega_t^n)^{-1} \text{diag}(\chi)\delta \left(\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} (1 + \hat{y}_t^{h\top} \frac{\text{diag}(\chi)\delta}{\lambda})^{-\gamma} - 1 \right), \end{aligned} \quad (39)$$

where $q^h(w_t^n; \theta)$ is the investor's marginal value of wealth given the hypothetical D^h (with ODE for $q^h(w_t^n; \theta)$ given in Appendix A.5.3), while $q^n(w_t^n)$ is the marginal value of wealth in

equilibrium with constant u^n . The implied demand slope is as follows:

$$\begin{aligned}
\mathcal{C}^{dyn}(w_t^n; \theta) = & \frac{[\hat{y}_t^h - \hat{y}_t^n]_i}{\delta_i} = \underbrace{\frac{1}{\gamma} [(\Omega_t^n)^{-1}]_{ii} + \frac{1}{\gamma} \frac{\sum_{j \neq i} [(\Omega_t^n)^{-1}]_{ij} [\delta]_j}{[\delta]_i}}_{\text{myopic slope}} \\
& + \underbrace{\frac{1}{\gamma} \left[\frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} - \frac{q^{n'}(w_t^n)}{q^n(w_t^n)} \right] \frac{[y_t^n w_t^n]_i}{[\delta]_i}}_{\text{Brownian hedging}} \\
& + \underbrace{\frac{1}{\gamma} \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \hat{y}_t^{h\top} \frac{\text{diag}(\chi) \delta}{\lambda} \right)^{-\gamma} - 1 \right] \frac{[(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]_i}{[\delta]_i}}_{\text{hedging against reversal}}. \tag{40}
\end{aligned}$$

Dynamic slope versus myopic slope. Despite the conceptual differences, our concepts of myopic and dynamic slopes are intimately related. The next proposition states that the dynamic concept is identical to the myopic one if the return path changes only for an instant.

Proposition 7. *For CRRA investors, the limit of the dynamic slope is the myopic slope when the hypothetical dividend path is perfectly transitory:*

$$\lim_{\lambda \rightarrow \infty} \mathcal{C}^{dyn}(w_t^n; \theta) = \mathcal{C}^{myo}(w_t^n; \delta).$$

Figure 2 illustrates the differences between the myopic and dynamic slopes, under the calibrated parameters (see the details of calibration in Appendix E.3). The blue solid line plots how much the investor would change her demand as a function of the given change in instantaneous return δ in our dynamic setting. Hence, dynamic slope, $\mathcal{C}^{dyn}(w_t^n; \theta)$ approximates the slope of this curve. We also plot the gray line as the corresponding myopic component given in (40); and the red dash-dotted line which gives the myopic demand plus the Brownian hedging component. Note that, just like the static demand curve $\mathcal{C}^{\text{stat}}(\delta)$ in (16), the myopic component is linear in δ as seen from (40).

There are two immediate takeaways from Figure 2: First, the change of asset demand is non-linear so that the demand curve turns flatter for larger shocks. Second, the curve tends to be flatter compared to the (linear) myopic demand; this is due to the classic Brownian and reversal hedging components à la Merton (1971), which, as Figure 2 shows, tends to be negative under our calibration.²⁷

Figure 2 also illustrates that as long as the shock to expected returns, δ , is not too large, the intertemporal hedging components tend to be small, hence the dynamic slope is close to its myopic component. This fact, and its simpler structure, might make the myopic slope an

²⁷For the interested reader, Appendix D discusses the economic intuition of why the Dynamic Slope tends to be flatter than its myopic counterpart.

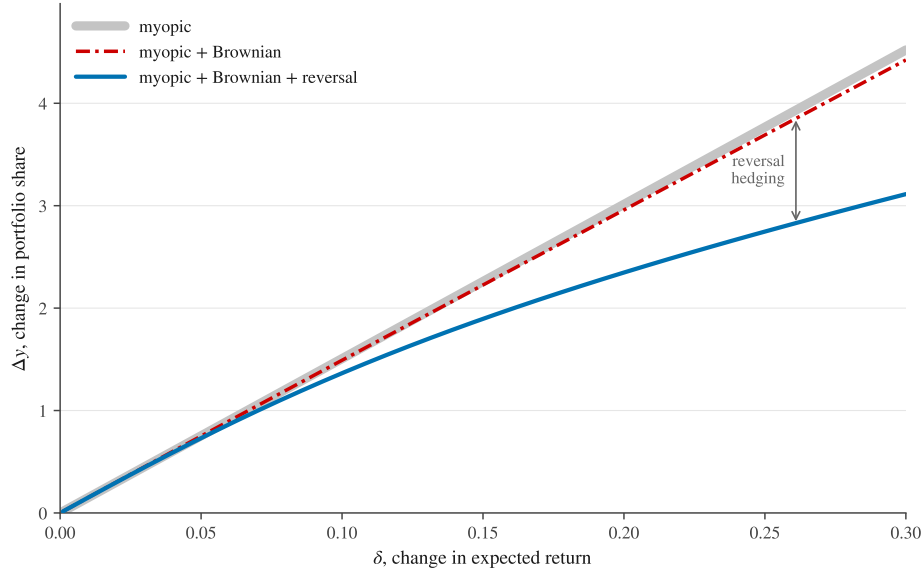


Figure 2: Change in demand as a function of change in expected return and its components. The blue solid line gives the investor's dynamic demand, which is the change in the investor's portfolio share Δy as a function of the change in the given instantaneous expected return δ . The gray line gives the myopic component, the red dash-dotted line gives the myopic demand plus the Brownian hedging component; so the gap between solid blue line and dash-dotted red line is the reversal hedging component. The parameter values are $\bar{D} = 0.415$, $\sigma = 0.57$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S = 1$, $u^n = 0$, $\lambda = 1$, and $\chi = 2/3$; the demand change is evaluated at the mean of the stationary distribution of investors' wealth, $\bar{w} = 5.03$.

attractive alternative to dynamic slope as a concept to capture investors' demand sensitivity. Indeed, we shall demonstrate soon, that whichever of them the econometrician wishes to recover based on shifts to residual supply, the problems remain the same.

It is important to emphasize that this does not imply that the effect of intertemporal hedging can be ignored. In terms of our minimal example, the quantitatively small intertemporal hedging corresponds to a small $a(\cdot)$. However, as the last term in (3) illustrates, if the investor's demand curve shifts, the mismeasurement of its slope is not proportional to the absolute size of $a(\cdot)$; it instead depends on the ratio between the shift size and the corresponding change in expected return, $\frac{a(y^s) - a(y^n)}{\mathbb{E}(R^s) - \mathbb{E}(R^n)}$. We will demonstrate below that this ratio can be large, even when the difference between dynamic slope and myopic slope is small.

5.3 Measured Slope versus Its Conceptual Counterparts

Per Definition 1, the measured slope of asset i is simply the ratio of the change in the representative investor's portfolio share $[y^s - y^n]_i$ to the change in its expected return $[\Delta \mathbb{E}_t(R)]_i$ as the equilibrium responds to a one-off residual supply shock. With CRRA utility, the optimal portfolio policy (33) implies that the measured slope is

$$\begin{aligned}
\mathcal{M}(w_t^n; \lambda, u^n, u^s) = & \underbrace{\mathcal{C}^{\text{myo}}(w_t^n; \delta)|_{\delta=\Delta\mathbb{E}_t(R)} + \frac{1}{\gamma} \frac{[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}] (\mathbb{E}_t(R^s) - r\mathbf{1})_i}{[\Delta\mathbb{E}_t(R)]_i}}_{\text{myopic measured slope}} + \\
& + \underbrace{\frac{1}{\gamma} \left[\frac{q^{s'}(w_t^s)}{q^s(w_t^s)} w_t^s y_t^s - \frac{q^{n'}(w_t^n)}{q^n(w_t^n)} w_t^n y_t^n \right]_i}_{\text{measured Brownian hedging}} + \underbrace{\frac{\lambda}{\gamma} \left[(\Omega_t^s)^{-1} \left(\frac{q^n(w_t^s(1+y_t^{s\top} J_{Rt}^s))}{q^s(w_t^s)[1+\hat{y}_t^{s\top} J_{Rt}^s]^\gamma} - 1 \right) J_{Rt}^s \right]_i}_{\text{measured hedging against reversal}}.
\end{aligned} \tag{41}$$

For easy comparison with the Dynamic slope, we add “measured” to the label of each component.

Now we are ready to confront the main question of our paper in the general setting: to what extent can an econometrician with an instrument capturing a shift in the residual supply curve successfully measure the slope of an investor’s demand curve? Under what conditions does the measured slope recover its conceptual counterparts, the Dynamic slope or the Myopic slope?

Measured slope versus myopic slope. Note that Expression (41) immediately provides an intuitive decomposition for the difference between measured and myopic slope, $\mathcal{M}(w_t^n; \lambda, u^n, u^s) - \mathcal{C}^{\text{myo}}(w_t^n; \delta)|_{\delta=\Delta\mathbb{E}_t(R)}$. The second term in (41) is the endogenous risk effect, while the third and fourth terms are the effect of amplified intertemporal hedging and measured hedging against reversal effects.

Measured slope versus dynamic slope. Now we turn to the comparison of $\mathcal{M}$ to $\mathcal{C}^{\text{dyn}}$. To give this exercise a fair chance, we pick i) $\delta = \Delta\mathbb{E}_t(R)$ to match the size and direction of the expected return shock due to the shift in residual supply; ii) $\text{diag}(\chi)\delta = \lambda J_{Rt}^s$ to match the composition; and iii) λ to match the persistence of the shock.

We now show that even if we match the parameters of these objects as closely as possible, $\mathcal{C}^{\text{dyn}}$ and $\mathcal{M}$ in general differ because $\mathcal{M}$ takes into account the adjustment of the dynamic equilibrium to the residual supply shock. Just as before, we can trace back the difference between $\mathcal{C}^{\text{dyn}}$ and $\mathcal{M}$ to three distinct sources.

Endogenous risk effect. First, recall that the myopic component of the slope of demand, which is the product of risk-tolerance and the inverse of the covariance matrix of returns Ω_t , is analogous to the static case. As the form (29) illustrates, both of these elements are equilibrium objects for general utility functions. In our CRRA setting, (32) implies the risk-tolerance to be an exogenous parameter $\frac{V_{\hat{w}}(\cdot)}{\hat{w}(-V_{\hat{w}}(\cdot))} \equiv \frac{1}{\gamma}$. However, the covariance matrix

of returns is endogenous because the shift in residual supply can change the riskiness of the asset's endogenous price and its covariance with other assets, even if cash-flows do not change.

As the slope of the dynamic demand curve is conceptually measuring how a single agent reacts to changes in returns, our measure $\mathcal{C}^{\text{dyn}}$ is not affected by these general equilibrium effects. Therefore, the myopic component of $\mathcal{C}^{\text{dyn}}$ will differ from that of $\mathcal{M}$. The following corollary states the result, obtained by differencing the myopic components of $\mathcal{M}$ in (41) and $\mathcal{C}^{\text{dyn}}$ in (40).

Corollary 1. *For a CRRA or log investor, the myopic slope differs from myopic measured slope because the return covariance matrix changes, i.e. there is an endogenous risk effect:*

$$\text{myopic measured slope} - \text{myopic slope} = \frac{1}{\gamma} \frac{[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}] (\mathbb{E}_t(R^s) - r\mathbf{1})_i}{[\Delta \mathbb{E}_t(R)]_i}; \quad (42)$$

where $\gamma = 1$ for the log investor.

As we discussed in Section 4 right after Proposition 2, the economic intuition behind the endogenous risk effect is closely related to standard amplification effects in heterogeneous agent models. Ω_t^s differs from Ω_t^n because all investors increase their holdings of the asset when hedgers liquidate following the shock. The investors' increased exposure to the fundamental dividend shocks tends to imply an increased volatility ((27)) and increased comovement with the other assets (Proposition 3). Hence, given the inverse operator, these effects tend to be negative, making the measured slope flatter than the dynamic slope.

Note the similarity of (42) with the second term in (3) which we labeled as the endogenous risk effect due to the tilt of the demand curve. Just as in our minimal example, the size of this effect depends on the ratio of the tilt, i.e., the change in the (inverse) covariance matrix and the corresponding change in expected returns. Both are endogenously determined. Because of this, as we will see in our calibrated examples, the endogenous risk effect could be large even if the change in covariance matrix (the numerator) is relatively small.

The effect of amplified intertemporal hedging. We now move on to the intertemporal hedging components, which also differ between dynamic slope and measured slope. In particular, the difference of intertemporal hedging with respect to the Brownian shock in (40) and (41) gives the expressions in the next corollary, where the common $\frac{q^{n'}(w_t^n)}{q^n(w_t^n)}$ terms cancel.

Corollary 2. *The amplified intertemporal hedging effect is:*

$$\begin{aligned} & \text{measured Brownian hedging in } \mathcal{M} - \text{Brownian hedging in } \mathcal{C}^{\text{dyn}}|_{\delta=\Delta \mathbb{E}_t(R), \text{diag}(\chi)\delta=\lambda J_{Rt}^s} = \\ &= \frac{1}{\gamma} \frac{\frac{q^{s'}(w_t^s)}{q^s(w_t^s)} [w_t^s y_t^s]_i - \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} [w_t^n y_t^n]_i}{[\Delta \mathbb{E}_t(R)]_i} \end{aligned} \quad (43)$$

for the Case 2 (CRRA investors) and 0 for Case 1 (log investors).

There are two sources of this difference. The first is that the shapes of the value functions under the hypothetical and under the stressed state, that is, $q'(\cdot)/q(\cdot)$, are different. This effect tends to be quantitatively small in our calibration exercises, since in both cases the value functions respond to a similar change in expected returns.

The second difference is more quantitatively important. Unlike $\mathcal{C}^{\text{dyn}}$, the general equilibrium effect enters $\mathcal{M}$: in equilibrium, all investors react to the shift in residual demand, and therefore, aggregate positions change from $w_t^n y_t^n$ to $w_t^s y_t^s$. Each investor understands that if others have a larger position of the asset that will increase the covariance of her wealth with future investment opportunities. This is so, because the portfolio of investors as a group performs well, the resulting increase in aggregate wealth implies larger risk-bearing capacity, lower risk premia, therefore, worse investment opportunities. An investor with $\gamma > 1$ prefers assets which are a good hedge against fluctuating investment opportunities. Hence, a larger component of y implies a larger component of $\hat{y}$. (An investor with $\gamma < 1$ prefers to cut back on such assets to have more funds to invest when others have lower risk-bearing capacity, hence investment opportunities are better.)

Linking the above observation back to our minimal example in Section 2, this is equivalent to having an endogenous intercept a^ξ that is increasing in investors' aggregate risk exposure y . Hence, $a^s > a^n$, so the investor's demand curve shifts upward.²⁸ We call it the “amplified intertemporal hedging effect” and, for $\gamma > 1$, it makes the measured slope steeper than the dynamic slope, going against the endogenous risk effect.

Hedging against reversal effect. Finally, noticing that $\lambda J_{Rt}^s = \text{diag}(\chi) \Delta \mathbb{E}_t(R)$, the next corollary gives the last difference between $\mathcal{C}^{\text{dyn}}$ and $\mathcal{M}$ due to intertemporal hedging demand for the reversal. Again, the difference comes from the fact that our measured slope is affected by the equilibrium change in all price paths due to the supply shock.

Corollary 3. *The hedging effect against the reversal for CRRA investors is:*

$$\begin{aligned} & \text{hedging against reversal in } \mathcal{M} - \text{hedging against reversal in } \mathcal{C}^{\text{dyn}} \big|_{\delta = \Delta \mathbb{E}_t(R), \text{diag}(\chi)\delta = \lambda J_{Rt}^s} = \\ &= \frac{\frac{1}{\gamma} \left[\left\{ (\Omega_t^s)^{-1} \left[\frac{q^n(w_t^s(1+y_t^{s\top} J_{Rt}^s))}{q^s(w_t^s)(1+\hat{y}_t^{s\top} J_{Rt}^s)} - 1 \right] - (\Omega_t^n)^{-1} \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)(1+\hat{y}_t^{h\top} \frac{\text{diag}(\chi)\delta}{\lambda})^\gamma} - 1 \right] \right\} \text{diag}(\chi) \Delta \mathbb{E}_t(R) \right]_i}{[\Delta \mathbb{E}_t(R)]_i}. \end{aligned} \quad (44)$$

For the log investors $\gamma = 1$ and $q^h(\cdot) = q^n(\cdot) = q^s(\cdot) \equiv 1$.

²⁸This general equilibrium effect is somewhat similar to the standard aggregate demand effect in macroeconomic models; for instance, in the Marginal Propensity to Consume literature, one dollar of additional individual consumption will translate to some endogenous increase in individual wealth, which induces more consumption.

It is clear that shocks to residual supply exactly recover the slope of CRRA investors' demand only if the sum of the above three objects—endogenous risk effect, amplified intertemporal hedging effect, and hedging against reversal effect—is exactly zero.

In the next subsection, through two applications, we demonstrate that there is no reason that any of these terms should be negligible under reasonable parameter values—to the contrary, they can be quantitatively substantial. Hence, generically, we should not expect that shifts in residual supply recover the slope of dynamic demand.

6 Applications and Discussion

In this section we perform a quantitative study on the difference between the measured slope and its conceptual counterpart, with the help of two applications. We then discuss the connection between our model and the reduced-form factor-based asset demand system in HHHKL, and the way forward given our theoretical results regarding the wedge between these two objects.

6.1 Two Applications: Factor Elasticity and Micro Elasticity

We develop two applications to illustrate our results. These examples are deliberately set in markedly different contexts to highlight the generality of our quantitative findings.

Factor elasticity. Our first calibration focuses on the effect of a relative demand shock on a few large factors. To this end, we calibrate to the study of Ben-David et al. (2021) focusing on factor-level elasticities. Our baseline parameters are obtained by calibrating our model to asset return characteristics and fund flow patterns in Ben-David et al. (2021). They show that when investment styles are highly correlated with Morningstar ratings, changes in fund ratings result in style-level fund flows which lead to price effects. In our calibration, we set $I = 2$. We interpret the two assets as two portfolios of different investment styles. In particular, we consider the first asset as the small value portfolio and the second asset as the large growth portfolio. Our choice of parameters is summarized in Table 1 Panel A. For instance, the persistence is 2 in our factor elasticity calibration, matching the approximate mean duration of fund flows²⁹.

Micro elasticity. In the second application, we are interested in index inclusion so that the residual supply shock affects only a few assets out of many, by calibrating our model to the recent study of Greenwood and Sammon (2024). Similar to our multi-asset setting in Section 4.3, we picture a market with a large number of assets with the same cash-flow risk

²⁹See Ben-David et al. (2021, Figure 5).

(A) Factor elasticity			
Para.	Description	Value	Targeted Moments
$\bar{D}$	Expected cash flow of assets	[0.27, 0.26]	historical returns
σ	Cash flow volatility	diag(0.485, 0.485)	historical return volatilities
α	Hedgers' absolute risk aversion	1	normalization
γ	Investors' relative risk aversion	3	literature
r	Risk-free rate	0.03	historical average interest rate
ρ	Investors' time discount rate	0.08	investor holdings
S	Asset supply	[1, 1]	normalization
u^n	Hedgers' endowment in normal state	[0, 0]	normalization
u^s	Hedgers' endowment in stressed state	[0.15, 0]	fund flows Ben-David et al. (2021)
λ	Intensity of state transitions	0.5	persistence Ben-David et al. (2021)

(B) Micro elasticity			
Para.	Description	Value	Targeted Moments
D_i	Expected cash flow of asset i	0.0369	historical returns
σ_i	Cash flow volatility of asset i	0.0692	historical return volatilities
κ	Cash flow correlation of i and j	0.3	idiosyncratic volatility
α	Hedgers' absolute risk aversion	1	normalization
γ	Investors' relative risk aversion	3	literature
r	Risk-free rate	0.03	historical average interest rate
ρ	Investors' time discount rate	0.08	investor holdings
S_i	Asset supply of asset i	0.0446	normalization ($1/\sqrt{I}$)
v_1	Endowment to index assets	0	normalization
v_2	Endowment to non-index assets	0.0107	fund flows GS (2024)
λ	Intensity of state transitions	0.2	persistence GS (2024)

Table 1: Parameters and Calibration Details. The table reports the parameter values used in the calibration exercises. Panel (A) presents the parameters for the factor elasticity calibration, while Panel (B) shows the parameters for the micro elasticity calibration. GS (2024) refers to Greenwood and Sammon (2024).

with the only difference in hedgers' endowment. In particular, suppose that in the normal state, hedgers' endowment for the first I_1 assets differs from the rest:

$$[u^n]_{i=1}^{I_1} = v_1, [u^n]_{i=I_1+1}^I = v_2.$$

where $v_1 \neq v_2$ are scalars. Our interpretation is that the first I_1 assets are in an index. We are interested in a shock for which $[u^n]_{i=2}^{I-1} = [u^s]_{i=2}^{I-1}$ while $[u^s]_1 = v_2$, $[u^s]_I = v_1$. That is, the first asset is excluded from and the last asset is included in the index. Just as in the example of Section 4, we intentionally design this example, so that the shock itself has a minimal effect on the structure of the economy—effectively, we only swap the labels of the first and the last assets. However, the main difference here is that instead of focusing on a diminishingly small, but infinitely persistent shock, we calibrate its size and persistence

(and each of the other parameters) to a real-world example. Our choice of parameters is summarized in Table 1 Panel B; for instance, we set the persistence to be 5 ($\lambda = 0.2$), which targets the conditional probability of index inclusion or deletion. For detailed explanations of our calibration exercises, see Appendices E.1 and E.2.³⁰

What do we learn from the two calibrated applications? Table 2 reports our main calibration results. Panel A is for factor elasticity calibration while Panel B is for micro elasticity. In each panel the colored (with bold font) column contains our benchmark parameterization, while the other columns show how our effects vary with the persistence of the demand shift.³¹ For each set of parameters, we calculate the myopic slope, the dynamic slope and the measured slope, and decompose the difference into our three effects.

There are a number of common takeaways across the two calibrations. First, while the dynamic slope is always smaller than the myopic slope, we find that the difference is quantitatively small. This is consistent with the numerical illustration in Figure 2. That is, whether one considers the myopic or the dynamic slope as the more intuitive measure for investor demand elasticity seems to have little practical significance.

Second, in all our examples, a shift in residual supply identifies a significantly flatter demand curve than investors' dynamic slope. In our baseline parameterizations, the measured slope is approximately 40% of the dynamic (and myopic) slope. This is because, just as in our minimal demand system, a shock that shifts residual supply also changes the shape of investors' demand function as general equilibrium effects alter the covariance of returns and induce changes in hedging demand. The fraction decreases with the persistence of the identifying shock (shrinking to 14% in our first example), meaning that the empirical underestimation of our conceptual demand slope is more pronounced when the identifying shock

³⁰In Appendix E, we rescale measured slope, $\mathcal{M}$, to make it comparable to standard estimates of price elasticity as in Gabaix and Koijen (2022). Under our baseline parameters, our factor elasticity calibration implies a price elasticity of 1.4. Ben-David et al. (2021) find 0.2. The index-inclusion calibration yields an implied price elasticity of 399, which is substantially lower than the commonly cited value of around 6250 from Petajisto (2009), but still far above the empirical estimate of 2.7 reported by Greenwood and Sammon (2024). On this gap, in contrast to the inelastic financial market hypothesis of Gabaix and Koijen (2022), Berk and Van Binsbergen (2025) argue that the classic CAPM framework is consistent with the observed index inclusion effect once researchers properly account for the translation between price and return. To this end, they divide the observed price change upon index inclusion by a duration of 40 years, corresponding to the average dividend yield. While we agree with Berk and Van Binsbergen (2025) on the conceptual point, we contend that the relevant duration should instead reflect the typical horizon over which stocks rotate in and out of benchmark indices—roughly five years, according to Greenwood and Sammon (2024); Pavlova and Sikorskaya (2023). This shorter duration helps explain why our implied elasticity remains substantially higher than the 2.7 estimate. These results are in line with a broader observation that we do not dispute in this paper: that frictionless asset pricing models, in their simplest form, face challenges in rationalizing the empirically small price elasticity.

³¹For each column in Panel A, we recalibrate the parameters u^ξ to ensure that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.03$. For each column in Panel B, we calibrate the parameters ν_1, ν_2 to ensure that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.01$.

(A) Factor elasticity					
	Persistence ($1/\lambda$)				
	0.20	0.50	1.00	2.00	5.00
Myopic slope	3.19	3.19	3.20	3.21	3.23
Dynamic slope	3.18	3.18	3.18	3.17	3.17
Measured slope	2.14	1.95	1.67	1.24	0.46
Measured slope/Dynamic slope	67%	61%	53%	39%	14%
Dynamic minus measured (DmM)	1.04	1.23	1.50	1.94	2.72
Shock to risk/DmM	134%	127%	119%	112%	104%
Diff hedging brownian/DmM	-34%	-27%	-20%	-13%	-6%
Diff hedging reversal/DmM	0%	1%	1%	1%	2%
(B) Micro elasticity					
	Persistence ($1/\lambda$)				
	0.20	0.50	1.00	2.00	5.00
Myopic slope	9.15	8.69	8.48	8.03	7.85
Dynamic slope	9.14	8.69	8.48	8.03	7.85
Measured slope	6.93	6.21	5.59	4.64	3.44
Measured slope/Dynamic slope	76%	72%	66%	58%	44%
Dynamic minus measured (DmM)	2.21	2.47	2.89	3.39	4.41
Shock to risk/DmM	166%	153%	141%	129%	116%
Diff hedging brownian/DmM	-66%	-53%	-41%	-29%	-16%
Diff hedging reversal/DmM	0%	0%	0%	0%	0%

Table 2: Comparative statics with respect to persistence $1/\lambda$. The table shows comparative statics with respect to shock persistence $1/\lambda$, for both the factor elasticity calibration (Panel A) and the micro elasticity calibration (Panel B). For the factor elasticity calibration, the colored (with bold font) column contains our benchmark parameterization: $\bar{D} = [0.27, 0.26]$, $\sigma = \text{diag}(0.485, 0.485)$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S = [1, 1]$, $u^n = [0, 0]$, $u^s = [0.15, 0]$, $\lambda = 0.5$ and $w = 2.11$. The remaining columns show how our effects vary with the shock persistence, with u^s recalibrated such that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.03$. For the micro elasticity calibration, the colored (with bold font) column contains our benchmark parameterization: $\bar{D}_i = 0.0369$, $\sigma_i = 0.0692$, $\kappa = 0.3$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S_i = 0.0446$, $v_1 = 0$, $v_2 = 0.0107$, $\lambda = 0.2$ and $w = 4.25$. The remaining columns show how our effects vary with the shock persistence, with v_2 recalibrated such that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.01\%$.

is more persistent. This is intuitive as a more persistent shock should have a larger effect on the dynamic equilibrium. (However, recall that in Proposition 2 we show that the under-measurement persists even for purely transitory shocks.)

Third, the decomposition of the difference between the measured slope and its conceptual counterpart shows that the endogenous risk effect dominates the others. When hedgers demand less of the asset, investors have to hold a larger share. Therefore, a shock to the aggregate wealth of this sector will have a larger effect on its return. From the point of view of individual investors, the assets are getting riskier and covary more with other stocks which

decreases the slope of their demand. In fact, because the amplified intertemporal hedging component is going against the endogenous risk effect whenever $\gamma > 1$, in all the highlighted cases the endogenous risk effect is more than 100% of the total under-measurement.³² This share declines with the persistence of the shock ranging from 104% to 166% across our examples.

Fourth, as intertemporal hedging is amplified in general equilibrium, this term plays a significant role in explaining the difference between dynamic slope and measured slope. We observe that while the direction of the effect is in line with the direction of Merton's intertemporal hedging, it is several orders of magnitude higher once the general equilibrium effect is taken into account. Recall from expression (43) that the size of this effect is mainly driven by the ratio of the shift in the equilibrium aggregate position to the shift in expected returns: $\frac{w^s y^s - w^n y^n}{[\Delta \mathbb{E}_t(R)]_i}$. In absolute terms, our results show that this effect can be responsible for up to 34% and 66% of the total mismeasurement in our two applications, respectively. Its absolute magnitude is especially large for less persistent shocks.

Finally, the component with respect to the intertemporal hedging for the reversal of the supply shock tends to be relatively small.

6.2 Price vs. Return Elasticity, Time-Variation vs. Shock-Dependence

It is useful to reemphasize two issues about the scope of our main insight. First, following Merton (1971), our framework links investors' portfolio share demand to asset returns, as in (33); however, our point applies equally to demand expressed in units of assets as a function of prices, the formulation most common in the empirical literature. Second, our point is not that the elasticity matrix is time-varying, but that it varies with the identifying shock itself. To show both, we rewrite our demand system in price–quantity units.

Let $\hat{Y}_t^\xi$ denote the individual investor's demand in units of assets—as opposed to the portfolio share $\hat{y}_t^\xi$ —when the economy is in state ξ , and let Y_t^ξ be the aggregate counterpart. Then, we can rewrite the optimal asset demand in (33) as

$$\underbrace{\hat{Y}_t^\xi}_{D_t} = \underbrace{\frac{-rw_t^\xi}{\gamma} \left[(\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \right]^{-1}}_{\mathcal{E}_t} P_t^\xi + e_t^\xi, \quad (45)$$

where the demand shifter e_t^ξ collects the remaining terms of the optimal demand; we spell it out in Appendix B.

The labeling in (45) shows how the dynamic demand curve maps to the standard formula-

³²In Appendix E, for both of our applications, we provide calibrations with $\gamma < 1$. In those cases, the endogenous risk effect is responsible for less than 100% of the total as our two main effects result in mismeasurements of the same sign.

tion of demand elasticity estimation in empirical work, $D_t = \mathcal{E}_t P_t + e_t$, where D_t is the vector of investors' demand, P_t is the price vector, $\mathcal{E}_t$ is the matrix of own- and cross-elasticities, and e_t is a vector of demand shifters (not necessarily zero-mean).

This mapping sharply illustrates the tension we are highlighting. First, $\mathcal{E}_t$ in (45) is a matrix of own- and cross-price elasticities, and its elements, just as e_t , are indexed by the state of residual supply ξ . An econometrician estimating price elasticities from residual-supply instruments therefore faces the same problem as one estimating return elasticities: the identifying shock moves the estimand and is correlated with the error term.

Second, the problem is not time-variation per se. If $\mathcal{E}_t$ fluctuated over time independently of the identifying shocks, a regression would still recover a well-defined average elasticity for the sample period, as in standard random-coefficient models. Ours is instead a correlated random coefficient problem: the coefficients of the demand curve co-move with the instrument, and averaging within the sample does not restore consistency.³³

6.3 The Way Forward

How do researchers move forward in light of these observations? We discuss several potential routes.

Finding small and transitory residual supply shocks? One possibility is to focus on economies where the mismeasurement is smaller. We show that if the shift in residual supply is small and transitory, the difference between measured slope and dynamic (or myopic) slope is significantly muted. However, as shown in Section 4, this approach cannot eliminate the problem because Proposition 2 demonstrates that the bias remains even in the limit.

Adding control variables or finding separate instruments? One question is whether carefully chosen controls can alleviate this problem. Recall the dynamic demand curve $\hat{y}_t^\xi = D(\mathbb{E}_t(R^\xi), \Xi_t^\xi)$ given in (4). Following Merton (1971), the second argument Ξ_t^ξ takes a concrete form, and for illustration consider one single risky asset so that $\Xi_t^\xi = ((\sigma_{Rt}^\xi)^2, IHD_t^\xi)$ are the return variance and the intertemporal hedging component. Suppose the researcher instruments $\Delta \mathbb{E}_t(R^\xi)$ and adds controls for the volatility of the asset along with a proxy for variation in intertemporal hedging. As we argued, if the instrument is based on a shift in residual supply, any candidate control variables for $(\sigma_{Rt}^\xi)^2$ and IHD_t^ξ are likely to be affected

³³This mapping also illustrates the complementarity between HHHKL and our work. HHHKL show that under the assumption of *homogeneous substitution conditional on observables*, cross-sectional and time-series information can be combined to estimate $\mathcal{E}_t$, given instruments that shift prices without affecting $\mathcal{E}_t$ or the unobserved e_t . In Appendix B we show that (45) locally has exactly this form—each asset has two endogenous characteristics, its aggregate holdings and its price loading on aggregate wealth—so the HHHKL procedure remains valid given such instruments. Finding such an instrument is itself challenging, however, because the endogenous risk emphasized by our dynamic model responds to the very shocks that move prices.

by the instrument itself. This leads to a bad control problem (Angrist and Pischke, 2009). The resulting endogeneity bias distorts the estimate of the slope with respect to expected returns. The canonical solution for this problem is to find separate instrument for each endogenous argument, that is, for $(\sigma_{Rt}^\xi)^2$ as well as for IHD_t^ξ . If such instruments satisfying the standard conditions could be found, the estimation would be consistent.³⁴

A structural approach to model equilibrium adjustment? Perhaps, a more promising avenue might be to impose more structure and separately model and instrument each argument of the demand function affected by the demand shift. Our fully solved model, especially the demand function of our CRRA investor characterized in (33) and the structure of the equilibrium covariance matrix (A.73) derived in Appendix A.1 offers some guidance along this direction.

For a simplistic example, the econometrician might be willing to make the assumption that the demand curve has the form of

$$\hat{g}_t^\xi = D_1 \underbrace{(\Omega_t^\xi)^{-1} \left(\mathbb{E}_t \left(R^\xi \right) - r\mathbf{1} \right)}_{\text{risk-adjusted instantaneous expected excess return}},$$

corresponding to the myopic version of the CRRA demand (33) with some coefficient D_1 . That is, for the investor, the risk-adjusted instantaneous expected excess return of each asset is now a sufficient statistic for the joint return distribution. If this is the case, then measuring the investors' response in demand to exogenous shocks to this object can consistently recover the demand curve. (The estimated coefficient D_1 corresponds to $1/\gamma$ under the CRRA specification, but could be interpreted broadly as the effective risk aversion for financial institutions, potentially reflecting limited capital or regulatory constraints rather than preferences.)

The trade-off is as usual: the structural approach requires taking a strong stance on the form of the market environment, together with each participant's incentives and constraints.

Why not focus on equilibrium measured slope $\mathcal{M}$? Another more reduced-form route is to step back from the view that asset markets can be seen as a demand system with classic supply and demand curves as defined in ECON101. For instance, if the Fed wants to know the effect on yields of buying 1 million USD worth of treasuries, they might not care whether the estimate corresponds to any investor's slope of demand. Perhaps the Fed only wants to know the resulting aggregate multiplier, taking into account the equilibrium mechanisms that tilt and shift investors' demand curves. If this is the case, we can take the measured relationship, akin to $\mathcal{M}(\cdot)$, between quantities and prices as the main object of interest. While, the

³⁴See the recent paper of Davis et al. (2026) constructing instruments for the variance argument as different combinations of past flow shocks. Their work provides preliminary evidence for the negative endogenous risk effect, in that they find a significant and economically sizable demand response to this argument.

estimated relationship would not correspond to the quantity response of a given investor group to a change in prices, it might provide sophisticated benchmarks for the literature on what “canonical” models imply on this aggregate multiplier and how sensitive it is to changes of parameters in the economy.

This perspective has a well-known precedent in macroeconomics: the literature on the government spending multiplier. It is by now well understood that the fiscal multiplier is not a structural parameter in the sense of Hurwicz (1962), that is, an object invariant to interventions. Instead, it is a reduced-form equilibrium response whose magnitude changes with the nature of the policy: the monetary stance, the financing scheme, and the persistence of the spending shock (Woodford, 2011; Christiano, Eichenbaum and Rebelo, 2011). In this sense, there is no “the” multiplier (Ramey, 2019), an instance of the critique of Lucas (1976). Importantly, the profession has not concluded that multiplier estimates are useless. Rather, as argued by Nakamura and Steinsson (2018), such estimates are best viewed as identified moments: causal effects of one particular intervention that do not transport to other policy environments, but that sharply discipline the set of admissible structural models. The measured slope $\mathcal{M}(\cdot)$ can play the same role in asset markets. An estimate obtained from index inclusion, say, is the equilibrium price response to that particular intervention, with its specific persistence and investor composition; extrapolating it to a different intervention, such as a large-scale asset purchase, requires a model. Its value lies in disciplining that model.

For this route, Appendix E provides some guidance. Taking the index inclusion effect studied in Section 6.1 as an example, we illustrate that the magnitude of $\mathcal{M}(\cdot)$ depends critically on two model ingredients: the calibrated persistence parameter of the demand shift induced by index inclusion or exclusion, λ , and the equilibrium wealth share of investors, w . The former is closely related to the duration effect emphasized by Davis et al. (2023) and van Binsbergen et al. (2025) (see also footnote 30), and the quantitative effect of λ is shown in Table 2. The latter is closely related to the intermediary asset pricing literature He and Krishnamurthy (2013); note that the calibration of w should also take into account that not all non-index investors trade actively, hence the effective wealth share of investors might be higher than what our baseline calibration suggests. We also refer interested readers to the calibrations in our companion paper, He et al. (2026), for a more detailed discussion.

7 Concluding Remarks

Measuring how investors would react to exogenous shifts in expected returns, or its mirror image, how returns react to changes in quantities, is crucial for a wide range of policy-relevant questions. Standard finance theory, however, implies that the slope of investors’ demand is an equilibrium object: it depends on the endogenous dynamics of future asset prices and, in particular, on the covariance structure of returns. We show that exogenous shocks to residual

supply, the instrument the empirical literature relies on to trace out demand, move exactly these objects. The shock tilts the demand curve through the endogenous risk effect and shifts it through the amplified intertemporal hedging effect, violating the exclusion restriction; neither effect vanishes when the shock is small or transitory.

Calibrating the model to a factor-elasticity and an index-inclusion context, we find that the measured slope is roughly 40% of its conceptual counterpart, with the bulk of the gap attributable to the endogenous risk effect. Demand curves may thus be considerably steeper than standard estimates suggest. Looking ahead, we see two promising routes: instrumenting each argument of the demand function separately, or stepping back from structural interpretation and treating the price response to supply shocks itself as the object of interest. In the latter case, the challenge to external validity has to be acknowledged and addressed: a route we pursue in the companion paper He et al. (2026).

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Appendix

A Proofs

Notation. The proofs distinguish several closely related quantities — three notions of investors’ holdings, hedgers’ holdings, and supply; we collect them here to fix notation.

Symbol	Object
y	investors’ demand, as a share of their wealth
z	investors’ wealth-scaled holding (demand per unit of wealth)
Y	investors’ holding, in units (used in the HHHKL comparison)
X	hedgers’ holdings, in dollar value
Q	hedgers’ holding, in units
S	asset supply (vector); in the symmetric example $S = s\mathbf{1}$, s scalar
Ω^ξ	covariance matrix of percentage (per-share) returns
Π^ξ	covariance matrix of dollar returns
J^ξ	jump in the price vector between states, $J_t^\xi = P_t^{-\xi} - P_t^\xi$
$\pi^\xi(w)$	risk-premium function, $P^\xi(w) = \bar{D}/r - \pi^\xi(w)$
x^n, x^s, x^h	normal, stressed, and hypothetical value of any variable x

A.1 Proof of Proposition 1 and Proposition 5

We first solve the hedger’s mean-variance problem, then set up the investor’s dynamic problem and its HJB equation, which yields the generic demand in terms of the value function. Specializing to log and CRRA utility in turn, we conjecture the value function, match the drift and diffusion (Itô) coefficients to pin down the marginal value of wealth, and impose market clearing to obtain the equilibrium price ODE and the demand decomposition. For the log utility case, we also specialize the demand to the $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ limits.

A.1.1 Hedgers’ Problem and Demand

We first consider a hedger’s problem. Let $Q_t^\xi \equiv \text{diag}(P_t^\xi)^{-1} X_t^\xi$ denote the hedger’s asset holdings in units when the economy is in state $\xi \in \{n, s\}$. From time t to $t + dt$, the hedger’s change in wealth is given by

$$dv_t^\xi = rv_t^\xi dt + Q_t^{\xi\top} (dD_t + dP_t^\xi - rP_t^\xi dt) + u^{\xi\top} dD_t.$$

Using (6) and (9), we have

$$\begin{aligned} dv_t^\xi = & \left[rv_t^\xi + Q_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) + u^{\xi\top} \bar{D} \right] dt \\ & + \left[(\sigma_{P_t}^\xi + \sigma) Q_t^\xi + \sigma u_t^\xi \right]^\top dB_t + Q_t^{\xi\top} J_t^\xi dN_t^\xi. \end{aligned}$$

Thus,

$$\frac{\mathbb{E}_t[dv_t^\xi]}{dt} = rv_t^\xi + Q_t^{\xi\top}(\bar{D} + \mu_{P_t}^\xi - rP_t^\xi + \lambda^\xi J_t^\xi) + u^{\xi\top} \bar{D}, \quad (\text{A.1})$$

$$\frac{\text{Var}_t(dv_t^\xi)}{dt} = [Q_t^{\xi\top}(\sigma_{P_t}^\xi + \sigma)^\top + u_t^{\xi\top} \sigma^\top] [(\sigma_{P_t}^\xi + \sigma)Q_t^\xi + \sigma u_t^\xi] + \lambda^\xi (Q_t^{\xi\top} J_t^\xi)^2. \quad (\text{A.2})$$

Substituting (A.1) and (A.2) into the hedger's mean-variance objective and taking first-order condition yields

$$\bar{D} + \mu_{P_t}^\xi - rP_t^\xi + \lambda^\xi J_t^\xi - \alpha \left\{ (\sigma_{P_t}^\xi + \sigma)^\top [(\sigma_{P_t}^\xi + \sigma)Q_t^\xi + \sigma u_t^\xi] + \lambda^\xi J_t^\xi J_t^{\xi\top} Q_t^\xi \right\} = 0.$$

Rearranging yields

$$Q_t^\xi = \left[(\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) + \lambda^\xi J_t^\xi J_t^{\xi\top} \right]^{-1} \left[\frac{\bar{D} + \mu_{P_t}^\xi - rP_t^\xi + \lambda^\xi J_t^\xi}{\alpha} - (\sigma_{P_t}^\xi + \sigma)^\top \sigma u_t^\xi \right]. \quad (\text{A.3})$$

Note that $\mu_{Rt}^\xi \equiv \text{diag}(P_t^\xi)^{-1}(\bar{D} + \mu_{P_t}^\xi)$, $\sigma_{Rt}^\xi \equiv (\sigma_{P_t}^\xi + \sigma) \text{diag}(P_t^\xi)^{-1}$, and $J_{Rt}^\xi \equiv \text{diag}(P_t^\xi)^{-1} J_t^\xi$. Then from (A.3), $X_t^\xi = \text{diag}(P_t^\xi) Q_t^\xi$ can be written as

$$X_t^\xi = \left[\Omega_t^\xi + \lambda^\xi J_{Rt}^\xi J_{Rt}^{\xi\top} \right]^{-1} \left[\frac{\mu_{Rt}^\xi - r\mathbf{1} + \lambda^\xi J_{Rt}^\xi}{\alpha} - (\sigma_{Rt}^\xi)^\top \sigma u_t^\xi \right],$$

where $\Omega_t^\xi \equiv (\sigma_{Rt}^\xi)^\top \sigma_{Rt}^\xi$.

A.1.2 Investors' Problem: HJB and Generic Demand

Let $\hat{w}_t^\xi$ denote an individual investor's time t wealth when the economy is in state ξ . Let $\hat{Y}_t^\xi \equiv \text{diag}(P_t^\xi)^{-1} \hat{w}_t^\xi \hat{y}_t^\xi$ denote the individual investor's corresponding asset holdings (in units). The investor's wealth $\hat{w}_t^\xi$ evolves according to

$$\begin{aligned} d\hat{w}_t^\xi &= r\hat{w}_t^\xi dt + \hat{Y}_t^{\xi\top} (dD_t + dP_t^\xi - rP_t^\xi dt) - \hat{c}_t^\xi dt \\ &= \left[r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{Y}_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) \right] dt + \hat{Y}_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top dB_t + \hat{Y}_t^{\xi\top} J_t^\xi dN_t^\xi. \end{aligned} \quad (\text{A.4})$$

Let w_t^ξ denote the aggregate wealth of investors at time t , and let Y_t^ξ denote the aggregate counterpart of $\hat{Y}_t^\xi$. Then w_t^ξ follows

$$dw_t^\xi = \left[rw_t^\xi - \hat{c}_t^\xi + Y_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) \right] dt + Y_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top dB_t + Y_t^{\xi\top} J_t^\xi dN_t^\xi. \quad (\text{A.5})$$

The individual investor's problem is to choose the optimal consumption and asset holdings $\{\hat{c}_t^\xi, \hat{Y}_t^\xi\}$ to maximize their discounted utility over intertemporal consumption, subject to the wealth dynamic (A.4) while taking prices P_t^ξ as given. Let $V^\xi(\hat{w}_t^\xi, w_t^\xi)$ denote the investor's

value function at time t in state ξ . The investor's HJB equation is

$$\begin{aligned} \rho V^\xi = \max_{\hat{c}_t^\xi, \hat{Y}_t^\xi} \left\{ u(\hat{c}_t^\xi) + V_{\hat{w}}^\xi \mu_t^{\hat{w}, \xi} + \frac{1}{2} V_{\hat{w}\hat{w}}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{\hat{w}, \xi} + V_w^\xi \mu_t^{w, \xi} + \frac{1}{2} V_{ww}^\xi (\sigma_t^{w, \xi})^\top \sigma_t^{w, \xi} \right. \\ \left. + V_{\hat{w}w}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{w, \xi} + \lambda^\xi (V^{-\xi, +} - V^\xi) \right\}, \end{aligned} \quad (\text{A.6})$$

where V^ξ stands for the value function $V^\xi(\hat{w}_t^\xi, w_t^\xi)$, while the superscript $+$ on $V^{-\xi, +}$ denotes that the value function is evaluated after the jump, that is

$$V^{-\xi, +} \equiv V^{-\xi} \left(\hat{w}_t^\xi + \hat{Y}_t^{\xi\top} J_t^\xi, w_t^\xi + Y_t^{\xi\top} J_t^\xi \right).$$

Moreover, $\mu_t^{\hat{w}, \xi}$ and $\sigma_t^{\hat{w}, \xi}$ are the drift and the diffusion of the individual investor's wealth $\hat{w}_t^\xi$, while $\mu_t^{w, \xi}$ and $\sigma_t^{w, \xi}$ are the drift and the diffusion of investors' aggregate wealth w_t^ξ . From (A.4) and (A.5),

$$\begin{aligned} \mu_t^{\hat{w}, \xi} &\equiv r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{Y}_t^{\xi\top} (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi), \\ \sigma_t^{\hat{w}, \xi} &\equiv (\sigma_{Pt}^\xi + \sigma) \hat{Y}_t^\xi, \\ \mu_t^{w, \xi} &\equiv r w_t^\xi - c_t^\xi + Y_t^{\xi\top} (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi), \\ \sigma_t^{w, \xi} &\equiv (\sigma_{Pt}^\xi + \sigma) Y_t^\xi. \end{aligned}$$

The first-order condition with respect to consumption implies that

$$u'(\hat{c}_t^\xi) = V_{\hat{w}}^\xi.$$

Similarly, the first-order condition with respect to asset holdings implies that $\hat{Y}_t^\xi$ satisfies

$$\begin{aligned} 0 = & V_{\hat{w}}^\xi (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi) + V_{\hat{w}\hat{w}}^\xi (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) \hat{Y}_t^\xi \\ & + V_{\hat{w}w}^\xi (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) Y_t^\xi + \lambda^\xi V_{\hat{w}}^{-\xi, +} J_t^\xi. \end{aligned} \quad (\text{A.7})$$

Rearranging the first-order condition (A.7), we can write the individual investor's asset demand as

$$\hat{Y}_t^\xi = \frac{V_{\hat{w}}^\xi}{(-V_{\hat{w}\hat{w}}^\xi)} (\Pi_t^\xi)^{-1} (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi) + \frac{V_{\hat{w}w}^\xi}{(-V_{\hat{w}\hat{w}}^\xi)} Y_t^\xi + \lambda^\xi \frac{V_{\hat{w}}^{-\xi, +}}{(-V_{\hat{w}\hat{w}}^\xi)} (\Pi_t^\xi)^{-1} J_t^\xi,$$

where we define $\Pi_t^\xi \equiv (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma)$. By definition, $\hat{y}_t^\xi = \frac{1}{\hat{w}_t^\xi} \text{diag}(P_t^\xi) \hat{Y}_t^\xi$ and $y_t^\xi = \frac{1}{w_t^\xi} \text{diag}(P_t^\xi) Y_t^\xi$. Then (A.4) can be rewritten as

$$\begin{aligned} d\hat{w}_t^\xi &= r\hat{w}_t^\xi dt + \hat{w}_t^\xi \hat{y}_t^{\xi\top} \left(dR_t^\xi - r\mathbf{1}dt \right) - \hat{c}_t^\xi dt \\ &= \left[(1 - \hat{y}_t^{\xi\top} \mathbf{1}) r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{w}_t^\xi \hat{y}_t^{\xi\top} \mu_{Rt}^\xi \right] dt + \hat{w}_t^\xi \hat{y}_t^{\xi\top} (\sigma_{Rt}^\xi)^\top dB_t + \hat{w}_t^\xi \hat{y}_t^{\xi\top} J_{Rt}^\xi dN_t^\xi. \end{aligned}$$

Similarly, (A.5) can be rewritten as

$$dw_t^\xi = \left[(1 - y_t^{\xi\top} \mathbf{1}) r w_t^\xi - c_t^\xi + w_t^\xi y_t^{\xi\top} \mu_{Rt}^\xi \right] dt + w_t^\xi y_t^{\xi\top} (\sigma_{Rt}^\xi)^\top dB_t + w_t^\xi y_t^{\xi\top} J_{Rt}^\xi dN_t^\xi.$$

And from (A.7), we have

$$\begin{aligned} \hat{y}_t^\xi &= \frac{V_{\hat{w}}^\xi}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1} (\mu_{Rt}^\xi - r \mathbf{1}) + \frac{V_{\hat{w}w}^\xi}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} w_t^\xi y_t^\xi + \lambda^\xi \frac{V_{\hat{w}}^{-\xi,+}}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1} J_{Rt}^\xi \\ &= \frac{V_{\hat{w}}^\xi}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1} \left(\mu_{Rt}^\xi - r \mathbf{1} + \lambda^\xi J_{Rt}^\xi \right) + \frac{V_{\hat{w}w}^\xi}{\hat{w}_t^\xi (-V_{\hat{w}\hat{w}}^\xi)} w_t^\xi y_t^\xi \\ &\quad + \frac{1}{\hat{w}_t^\xi} \frac{V_{\hat{w}}^{-\xi,+} - V_{\hat{w}}^\xi}{(-V_{\hat{w}\hat{w}}^\xi)} (\Omega_t^\xi)^{-1} \lambda^\xi J_{Rt}^\xi. \end{aligned}$$

A.1.3 Equilibrium ODE System and Demand: Log Investors

A log investor maximizes

$$\mathbb{E}_t \left[\int_t^\infty e^{-\rho(\tau-t)} \log \hat{c}_\tau d\tau \right], \quad (\text{A.8})$$

subject to the wealth dynamic (A.4). To facilitate solution, we let $\hat{z}_t^\xi \equiv \hat{Y}_t^\xi / \hat{w}_t^\xi$ and $z_t^\xi \equiv Y_t^\xi / w_t^\xi$. Then the investor's wealth dynamic (A.4) becomes

$$d\hat{w}_t^\xi = r \hat{w}_t^\xi dt + \hat{w}_t^\xi \hat{z}_t^{\xi\top} (dD_t + dP_t^\xi - r P_t^\xi dt) - \hat{c}_t^\xi dt. \quad (\text{A.9})$$

Substituting (6) and (9) into the individual investor's wealth dynamic (A.9) yields

$$\begin{aligned} d\hat{w}_t^\xi &= \left[r \hat{w}_t^\xi - \hat{c}_t^\xi + \hat{w}_t^\xi \hat{z}_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right] dt + \hat{w}_t^\xi \hat{z}_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top dB_t \\ &\quad + \hat{w}_t^\xi \hat{z}_t^{\xi\top} J_t^\xi dN_t^\xi. \end{aligned}$$

Similarly, the investors' total wealth dynamic aggregates to

$$\begin{aligned} dw_t^\xi &= \left[r w_t^\xi - c_t^\xi + w_t^\xi z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right] dt + w_t^\xi z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top dB_t \\ &\quad + w_t^\xi z_t^{\xi\top} J_t^\xi dN_t^\xi. \end{aligned}$$

The investor's HJB equation is

$$\begin{aligned} \rho V^\xi &= \max_{\hat{c}_t^\xi, \hat{z}_t^\xi} \left\{ \log \hat{c}_t^\xi + V_{\hat{w}}^\xi \hat{w}_t^{\hat{w},\xi} + \frac{1}{2} V_{\hat{w}\hat{w}}^\xi (\sigma_t^{\hat{w},\xi})^\top \sigma_t^{\hat{w},\xi} + V_w^\xi \mu_t^{w,\xi} + \frac{1}{2} V_{ww}^\xi (\sigma_t^{w,\xi})^\top \sigma_t^{w,\xi} \right. \\ &\quad \left. + V_{\hat{w}w}^\xi (\sigma_t^{\hat{w},\xi})^\top \sigma_t^{w,\xi} + \lambda^\xi (V^{-\xi,+} - V^\xi) \right\}, \end{aligned} \quad (\text{A.10})$$

where

$$\mu_t^{\hat{w},\xi} = r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{w}_t^\xi \hat{z}_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi), \quad (\text{A.11})$$

$$\begin{aligned} \sigma_t^{\hat{w},\xi} &= (\sigma_{P_t}^\xi + \sigma) \hat{z}_t^\xi \hat{w}_t^\xi, \\ \mu_t^{w,\xi} &= r w_t^\xi - c_t^\xi + w_t^\xi z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi), \\ \sigma_t^{w,\xi} &= (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi. \end{aligned} \quad (\text{A.12})$$

We conjecture (and later verify) that the investor's value function takes the form

$$V^\xi(\hat{w}_t^\xi, w_t^\xi) = \frac{1}{\rho} \log \hat{w}_t^\xi + \bar{q}^\xi(w_t^\xi). \quad (\text{A.13})$$

Substituting (A.13) into (A.10) yields

$$\begin{aligned} &\log \hat{w}_t^\xi + \rho \bar{q}^\xi(w_t^\xi) = \\ &\max_{\hat{c}_t^\xi, \hat{z}_t^\xi} \left\{ \log \hat{c}_t^\xi + \frac{1}{\rho \hat{w}_t^\xi} \mu_t^{\hat{w},\xi} - \frac{1}{2\rho(\hat{w}_t^\xi)^2} (\sigma_t^{\hat{w},\xi})^\top \sigma_t^{\hat{w},\xi} + \bar{q}^{\xi'}(w_t^\xi) \mu_t^{w,\xi} + \frac{1}{2} \bar{q}^{\xi''}(w_t^\xi) (\sigma_t^{w,\xi})^\top \sigma_t^{w,\xi} \right. \\ &\left. + \lambda^\xi \left[\frac{1}{\rho} \log \left(\hat{w}_t^\xi + \hat{w}_t^\xi \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi) \right) + \bar{q}^{-\xi} \left(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi) \right) - \frac{1}{\rho} \log \hat{w}_t^\xi - \bar{q}^\xi(w_t^\xi) \right] \right\}. \end{aligned} \quad (\text{A.14})$$

Taking the first-order condition with respect to consumption yields

$$\frac{1}{\hat{c}_t^\xi} - \frac{1}{\rho \hat{w}_t^\xi} = 0.$$

Thus,

$$\hat{c}_t^\xi = \rho \hat{w}_t^\xi. \quad (\text{A.15})$$

Taking the first-order condition with respect to the investor's wealth-scaled asset holdings implies that $\hat{z}_t^\xi$ satisfies

$$0 = (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) - (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \hat{z}_t^\xi + \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}.$$

Rearranging, we get

$$\bar{D} + \mu_{P_t}^\xi - rP_t^\xi = (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \hat{z}_t^\xi - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}. \quad (\text{A.16})$$

Then

$$\hat{z}_t^\xi = (\Pi_t^\xi)^{-1} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) + \lambda^\xi \frac{(\Pi_t^\xi)^{-1} (P_t^{-\xi} - P_t^\xi)}{1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}. \quad (\text{A.17})$$

From (A.17), since $\hat{y}_t^\xi = \text{diag}(P_t^\xi) \hat{z}_t^\xi$,

$$\begin{aligned}\hat{y}_t^\xi &= (\Omega_t^\xi)^{-1}(\mu_{Rt}^\xi - r\mathbf{1}) + \lambda^\xi \frac{(\Omega_t^\xi)^{-1} J_{Rt}^\xi}{1 + \hat{y}_t^{\xi\top} J_{Rt}^\xi} \\ &= (\Omega_t^\xi)^{-1} \left(\mu_{Rt}^\xi - r\mathbf{1} + \lambda^\xi J_{Rt}^\xi \right) + \lambda^\xi (\Omega_t^\xi)^{-1} \left(\frac{1}{1 + \hat{y}_t^{\xi\top} J_{Rt}^\xi} - 1 \right) J_{Rt}^\xi.\end{aligned}\tag{A.18}$$

Derivation of ODEs. Note that $\hat{z}_t^\xi = z_t^\xi$, $\hat{c}_t^\xi = c_t^\xi$ and $\hat{w}_t^\xi = w_t^\xi$ in equilibrium, then (A.16) implies

$$\bar{D} + \mu_{Pt}^\xi - rP_t^\xi = (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) z_t^\xi - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)},\tag{A.19}$$

and from (A.15), $c_t^\xi = \rho w_t^\xi$. Substituting into the HJB equation (A.14) yields

$$\begin{aligned}\rho \bar{q}^\xi(w_t^\xi) &= \log \rho + \frac{1}{\rho} \left[r - \rho + \hat{z}_t^{\xi\top} (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi) \right] \\ &\quad - \frac{1}{2\rho} \hat{z}_t^{\xi\top} (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) \hat{z}_t^\xi \\ &\quad + \bar{q}^{\xi'}(w_t^\xi) w_t^\xi \left[r - \rho + z_t^{\xi\top} (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi) \right] \\ &\quad + \frac{1}{2} \bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 z_t^{\xi\top} (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) z_t^\xi \\ &\quad + \lambda^\xi \left[\frac{1}{\rho} \log(1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)) + \bar{q}^{-\xi} \left(w_t^\xi + w_t^\xi \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi) \right) - \bar{q}^\xi(w_t^\xi) \right].\end{aligned}$$

Note that $\hat{w}_t^\xi$ has all canceled out. Thus, we have verified the conjectured value function (A.13). Using the fact that $\hat{z}_t^\xi = z_t^\xi$, we get

$$\begin{aligned}\rho \bar{q}^\xi(w_t^\xi) &= \log \rho + \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] \left[r - \rho + z_t^{\xi\top} (\bar{D} + \mu_{Pt}^\xi - rP_t^\xi) \right] \\ &\quad + \frac{1}{2} \left[\bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 - \frac{1}{\rho} \right] z_t^{\xi\top} (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) z_t^\xi \\ &\quad + \lambda^\xi \left[\frac{1}{\rho} \log(1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)) + \bar{q}^{-\xi} \left(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi) \right) - \bar{q}^\xi(w_t^\xi) \right].\end{aligned}$$

Using (A.19),

$$\begin{aligned}
\rho \bar{q}^\xi(w_t^\xi) &= \log \rho + \left[\bar{q}'^\xi(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] \left[r - \rho + z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \right] \\
&\quad - \left[\bar{q}'^\xi(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \\
&\quad + \frac{1}{2} \left[\bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 - \frac{1}{\rho} \right] z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\
&\quad + \lambda^\xi \left[\frac{1}{\rho} \log(1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)) + \bar{q}^{-\xi} \left(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi) \right) - \bar{q}^\xi(w_t^\xi) \right].
\end{aligned}$$

Collecting terms, we get

$$\begin{aligned}
\rho \bar{q}^\xi(w_t^\xi) &= \log \rho + \left[\bar{q}'^\xi(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] (r - \rho) \\
&\quad + \left[\bar{q}'^\xi(w_t^\xi) w_t^\xi + \frac{1}{2} \bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 + \frac{1}{2\rho} \right] z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\
&\quad + \lambda^\xi \left[\frac{1}{\rho} \log(1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)) + \bar{q}^{-\xi} \left(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi) \right) - \bar{q}^\xi(w_t^\xi) \right. \\
&\quad \left. - \left(\bar{q}'^\xi(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right) \frac{z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right]. \tag{A.20}
\end{aligned}$$

Substituting the investors' aggregate consumption c_t^ξ into the total wealth dynamic for investors, we get

$$dw_t^\xi = w_t^\xi \left[r - \rho + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right] dt + w_t^\xi z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top dB_t + w_t^\xi z_t^{\xi\top} J_t^\xi dN_t^\xi.$$

The investors' total wealth w_t^ξ and the state of the economy ξ are the two state variables in the model. Thus, the risky assets' prices in state ξ must be functions of the investors' total wealth in the state, that is $P_t^\xi \equiv P^\xi(w_t^\xi)$. Let

$$\tilde{\mu}_t^{w,\xi} \equiv w_t^\xi \left[r - \rho + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right], \tag{A.21}$$

$$\tilde{\sigma}_t^{w,\xi} \equiv (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi. \tag{A.22}$$

By Itô's lemma, we have

$$\begin{aligned}
dP_t^\xi &= \left[\tilde{\mu}_t^{w,\xi} P^{\xi'}(w_t^\xi) + \frac{1}{2} (\tilde{\sigma}_t^{w,\xi})^\top \tilde{\sigma}_t^{w,\xi} P^{\xi''}(w_t^\xi) \right] dt + P^{\xi'}(w_t^\xi) (\tilde{\sigma}_t^{w,\xi})^\top dB_t \\
&\quad + (P_t^{-\xi} - P_t^\xi) dN_t^\xi.
\end{aligned}$$

From (A.21) and (A.22), the drift of the risky assets' prices becomes

$$\begin{aligned}
& \tilde{\mu}_t^{w,\xi} P^{\xi'}(w_t^\xi) + \frac{1}{2} (\tilde{\sigma}_t^{w,\xi})^\top \tilde{\sigma}_t^{w,\xi} P^{\xi''}(w_t^\xi) \\
&= w_t^\xi [r - \rho + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi)] P^{\xi'}(w_t^\xi) + \frac{1}{2} z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\
&= \left[r - \rho + z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
&\quad + \frac{1}{2} z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\
&= \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
&\quad + z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right].
\end{aligned}$$

The diffusion of the risky assets' prices is

$$\tilde{\sigma}_t^{w,\xi} P^{\xi'}(w_t^\xi)^\top = (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi P^{\xi'}(w_t^\xi)^\top.$$

The risky assets' prices are conjectured to follow the Itô process (9). By matching coefficients, we get

$$J_t^\xi = P_t^{-\xi} - P_t^\xi,$$

while $\mu_{P_t}^\xi$ and $\sigma_{P_t}^\xi$ are respectively given by

$$\begin{aligned}
\mu_{P_t}^\xi &= \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
&\quad + z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right],
\end{aligned} \tag{A.23}$$

and

$$\sigma_{P_t}^\xi = (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi P^{\xi'}(w_t^\xi)^\top. \tag{A.24}$$

From (A.24),

$$(\sigma_{P_t}^\xi + \sigma) z_t^\xi = (\sigma_{P_t}^\xi + \sigma) z_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi + \sigma z_t^\xi.$$

Thus,

$$[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi] (\sigma_{P_t}^\xi + \sigma) z_t^\xi = \sigma z_t^\xi.$$

Then

$$(\sigma_{P_t}^\xi + \sigma) z_t^\xi = \frac{\sigma z_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}. \tag{A.25}$$

Equation (A.25) is the standard price–wealth feedback loop: a Brownian shock moves the price, hence investors’ wealth which hits the pricing kernel, hence the price again; the factor $1/(1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi)$ sums the resulting geometric series. Note that in any equilibrium with continuous P^ξ and finite return volatility, $1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi > 0$; this is because the expression equals one where the feedback vanishes (when $w_t \rightarrow \infty$, the model collapses to the frictionless Merton benchmark without endogenous risk, so asset prices equal $\bar{D}/r$; see Kondor and Vayanos (2019)), and in the relevant region of state space this expression cannot cross zero, since at that point (A.25) admits no solution.³⁵

Substituting (A.25) into (A.24) yields

$$\sigma_{Pt}^\xi = \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}.$$

Adding σ to both sides yields

$$\sigma_{Pt}^\xi + \sigma = \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi + [1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]\sigma}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}. \quad (\text{A.26})$$

Let

$$\hat{\sigma}_t^\xi \equiv \sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi + [1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]\sigma. \quad (\text{A.27})$$

Then

$$\sigma_{Pt}^\xi + \sigma = \frac{\hat{\sigma}_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}. \quad (\text{A.28})$$

Substituting (A.25) into (A.23) yields

$$\begin{aligned} \mu_{Pt}^\xi = & \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ & + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right]. \end{aligned}$$

Thus

$$\begin{aligned} & \bar{D} + \mu_{Pt}^\xi - r P_t^\xi \\ = & \bar{D} - r P_t^\xi + \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ & + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right]. \end{aligned}$$

³⁵Technically speaking, just as in Kondor and Vayanos (2019), we have not ruled out the existence of a solution where this expression is always negative; this class of solutions potentially corresponds to the divergent branch of the feedback loop, in which investors’ wealth would fall on good cash-flow news about their own portfolio.

From (A.19), and using (A.25) and (A.28), we have

$$\bar{D} + \mu_{P_t}^\xi - rP_t^\xi = \frac{\hat{\sigma}_t^{\xi\top} \sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}. \quad (\text{A.29})$$

Hence,

$$\begin{aligned} & \bar{D} - rP^\xi(w_t^\xi) + \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ & + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] \\ & = \frac{\hat{\sigma}_t^{\xi\top} \sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}. \end{aligned} \quad (\text{A.30})$$

The equation (A.30) follows from the investors' first-order condition and Itô's lemma. Moreover, from (A.29) we have

$$z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) = \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}.$$

Since there is a measure one of symmetric hedgers, the aggregate risky asset holdings by hedgers at time t in state ξ are given by

$$\begin{aligned} Q_t^\xi &= \left[(\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) + \lambda^\xi J_t^\xi J_t^{\xi\top} \right]^{-1} \left[\frac{\bar{D} + \mu_{P_t}^\xi - rP_t^\xi + \lambda^\xi J_t^\xi}{\alpha} - (\sigma_{P_t}^\xi + \sigma)^\top \sigma u^\xi \right] \\ &= \frac{1}{\alpha} \left\{ \frac{\hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \lambda^\xi (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top \right\}^{-1} \left\{ \lambda^\xi (P_t^{-\xi} - P_t^\xi) \right. \\ & \quad + \bar{D} - rP^\xi(w_t^\xi) + \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ & \quad \left. + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right\}. \end{aligned}$$

Market clearing requires that

$$Q_t^\xi + Y_t^\xi = Q_t^\xi + z_t^\xi w_t^\xi = S.$$

That is,

$$\begin{aligned}
& \frac{1}{\alpha} \left\{ \frac{\hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \lambda^\xi (P_t^{-\xi} - P_t^\xi)(P_t^{-\xi} - P_t^\xi)^\top \right\}^{-1} \left\{ \lambda^\xi (P_t^{-\xi} - P_t^\xi) \right. \\
& + \bar{D} - rP^\xi(w_t^\xi) + \left[r - \rho - \frac{\lambda^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \Big\} \\
& = S - z_t^\xi w_t^\xi.
\end{aligned} \tag{A.31}$$

The equation (A.31) follows from the market clearing condition.

When the economy switches from state ξ to $-\xi$ at time t , the investor's wealth jumps from w_t^ξ to $w_t^{-\xi}$ according to the following fixed point condition

$$w_t^{-\xi} = w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi). \tag{A.32}$$

Substituting (A.32) into (A.30) yields

$$\begin{aligned}
& \bar{D} - rP^\xi(w_t^\xi) + \left[r - \rho - \frac{\lambda^\xi \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{\frac{w_t^{-\xi}}{w_t^\xi}} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] \\
& = \frac{\hat{\sigma}_t^{\xi\top} \sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{\frac{w_t^{-\xi}}{w_t^\xi}}.
\end{aligned} \tag{A.33}$$

Furthermore, substituting (A.32) into (A.31) yields

$$\begin{aligned}
& \bar{D} - rP^\xi(w_t^\xi) + \left[r - \rho - \frac{\lambda^\xi \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{\frac{w_t^{-\xi}}{w_t^\xi}} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] \\
& = \alpha \left\{ \frac{\hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \lambda^\xi (P_t^{-\xi} - P_t^\xi)(P_t^{-\xi} - P_t^\xi)^\top \right\} (S - z_t^\xi w_t^\xi) \\
& + \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} - \lambda^\xi (P_t^{-\xi} - P_t^\xi).
\end{aligned} \tag{A.34}$$

Combining (A.33) and (A.34) yields

$$\begin{aligned}
& \frac{\hat{\sigma}_t^{\xi\top} \sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{\frac{w_t^{-\xi}}{w_t^\xi}} \\
&= \frac{\alpha \hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi (S - z_t^\xi w_t^\xi)}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \alpha \lambda^\xi (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top (S - z_t^\xi w_t^\xi) \\
&+ \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} - \lambda^\xi (P_t^{-\xi} - P_t^\xi).
\end{aligned}$$

Rearranging yields

$$\begin{aligned}
& \frac{(1 + \alpha w_t^\xi) \hat{\sigma}_t^{\xi\top} \sigma z_t^\xi - \alpha \hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi S}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \\
&= \lambda^\xi \left[\left(\frac{1}{\frac{w_t^{-\xi}}{w_t^\xi}} - 1 \right) (P_t^{-\xi} - P_t^\xi) + \alpha (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top (S - z_t^\xi w_t^\xi) \right]. \tag{A.35}
\end{aligned}$$

Note that

$$\hat{\sigma}_t^{\xi\top} \sigma z_t^\xi = w_t^\xi P^{\xi'}(w_t^\xi) z_t^{\xi\top} \Sigma z_t^\xi + [1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi] \Sigma z_t^\xi.$$

Hence, (A.33) simplifies to

$$\begin{aligned}
& \bar{D} - r P^\xi(w_t^\xi) + \left[r - \rho - \frac{\lambda^\xi \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{\frac{w_t^{-\xi}}{w_t^\xi}} \right] P^{\xi'}(w_t^\xi) w_t^\xi + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{2[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\
&= \frac{\Sigma z_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} - \frac{\lambda^\xi (P_t^{-\xi} - P_t^\xi)}{\frac{w_t^{-\xi}}{w_t^\xi}}. \tag{A.36}
\end{aligned}$$

Moreover, $\bar{q}^\xi(w_t^\xi)$ is the solution to the ODE (A.20), which can be rewritten as

$$\begin{aligned}
& \log \rho + \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] (r - \rho) - \rho \bar{q}^\xi(w_t^\xi) \\
&+ \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} \bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 + \frac{1}{2\rho} \right] \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \\
&+ \lambda^\xi \left[\frac{1}{\rho} \log \left(\frac{w_t^{-\xi}}{w_t^\xi} \right) + \bar{q}^{-\xi}(w_t^{-\xi}) - \bar{q}^\xi(w_t^\xi) - \left(\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right) \frac{\frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{\frac{w_t^{-\xi}}{w_t^\xi}} \right] = 0. \tag{A.37}
\end{aligned}$$

The investors' wealth-scaled asset holdings z_t^ξ are determined by (A.35), while prices

$P^\xi(w_t^\xi)$ and hedging demand $\bar{q}^\xi(w_t^\xi)$ are solutions to the ODEs (A.36) and (A.37).

Summary of the ODE system. Define

$$\pi^\xi(w_t^\xi) \equiv \frac{\bar{D}}{r} - P^\xi(w_t^\xi).$$

Then the investors' wealth-scaled asset holdings z_t^ξ satisfy

$$\begin{aligned} & \frac{(1 + \alpha w_t^\xi) \hat{\sigma}_t^{\xi\top} \sigma z_t^\xi - \alpha \hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi S}{[1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi} \\ &= \lambda^\xi \left[\left(\frac{w_t^\xi}{w_t^{-\xi}} - 1 \right) (P_t^{-\xi} - P_t^\xi) + \alpha (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top (S - z_t^\xi w_t^\xi) \right], \end{aligned}$$

where

$$\begin{aligned} \hat{\sigma}_t^\xi &= -w_t^\xi \sigma z_t^\xi \pi^{\xi'}(w_t^\xi)^\top + [1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi] \sigma, \\ P_t^{-\xi} - P_t^\xi &= \pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi}). \end{aligned}$$

The functions $\pi^\xi(w_t^\xi)$ and $\bar{q}^\xi(w_t^\xi)$ are solutions to the following ODEs

$$\begin{aligned} & r\pi^\xi(w_t^\xi) - \left[r - \rho - \frac{\lambda^\xi(w_t^{-\xi} - w_t^\xi)}{w_t^{-\xi}} \right] \pi^{\xi'}(w_t^\xi) w_t^\xi - \frac{z_t^{\xi\top} \Sigma z_t^\xi}{2[1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} \pi^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\ &= \frac{\Sigma z_t^\xi}{1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi} - \frac{\lambda^\xi [\pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi})]}{\frac{w_t^{-\xi}}{w_t^\xi}}, \end{aligned}$$

and

$$\begin{aligned} & \log \rho + \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] (r - \rho) - \rho \bar{q}^\xi(w_t^\xi) \\ &+ \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} \bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 + \frac{1}{2\rho} \right] \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} \\ &+ \lambda^\xi \left[\frac{1}{\rho} \log \left(\frac{w_t^{-\xi}}{w_t^\xi} \right) + \bar{q}^{-\xi}(w_t^{-\xi}) - \bar{q}^\xi(w_t^\xi) - \left(\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right) \frac{w_t^{-\xi} - w_t^\xi}{w_t^{-\xi}} \right] = 0. \end{aligned}$$

Moreover, the following fixed point condition must be satisfied

$$w_t^{-\xi} - w_t^\xi = w_t^\xi z_t^{\xi\top} [\pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi})].$$

The endogenous percentage return variance-covariance matrix is given by $\Omega_t^\xi \equiv (\sigma_{Rt}^\xi)^\top \sigma_{Rt}^\xi$,

where $\sigma_{Rt}^\xi \equiv (\sigma_{Pt}^\xi + \sigma) \text{diag}(P_t^\xi)^{-1}$. Then from (A.26),

$$\begin{aligned}\Omega_t^\xi &= \text{diag}(P_t^\xi)^{-1} (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) \text{diag}(P_t^\xi)^{-1} \\ &= \text{diag}(P_t^\xi)^{-1} \left[\sigma + \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right]^\top \left[\sigma + \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right] \text{diag}(P_t^\xi)^{-1}.\end{aligned}$$

Thus, the variance-covariance matrix exhibits a factor structure and depends on the endogenous state variable, the investors' total wealth w_t^ξ .

Limit cases with $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$. The log investor's optimal portfolio policy $\hat{y}_t^\xi$ is given by (A.18). Our setup focuses on $\lambda^n = 0$ and $\lambda^s = \lambda$. Since we define $\mathbb{E}_t(R^\xi) \equiv \mathbb{E}_t(dR_t^\xi)/dt = \mu_{Rt}^\xi + \lambda^\xi J_{Rt}^\xi$, we can then rewrite $\hat{y}_t^\xi$ as

$$\hat{y}_t^\xi = (\Omega_t^\xi)^{-1} \left(\mathbb{E}_t(R^\xi) - r\mathbf{1} \right) + \lambda^\xi (\Omega_t^\xi)^{-1} \left(\frac{1}{1 + \hat{y}_t^{\xi\top} J_{Rt}^\xi} - 1 \right) J_{Rt}^\xi. \quad (\text{A.38})$$

It is immediate that in state n ,

$$\hat{y}_t^n = (\Omega_t^n)^{-1} (\mathbb{E}_t(R^n) - r\mathbf{1}).$$

Similarly, when $\lambda \rightarrow 0$, the investor's demand in state s is

$$\lim_{\lambda \rightarrow 0} \hat{y}_t^s = (\Omega_t^s)^{-1} (\mathbb{E}_t(R^s) - r\mathbf{1}).$$

Move on to the second term which equals $-\frac{\hat{y}_t^{s\top} J_{Rt}^s}{1 + \hat{y}_t^{s\top} J_{Rt}^s} (\Omega_t^s)^{-1} (\lambda J_{Rt}^s)$ on the right-hand side of (A.38), because the equilibrium expected returns $\lambda J_{Rt}^s = \mathbb{E}_t(R^s) - \mu_{Rt}^s$ are bounded, we know that $J_{Rt}^s = (\lambda J_{Rt}^s)/\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$. The scalar prefactor therefore vanishes while $(\Omega_t^s)^{-1} (\lambda J_{Rt}^s)$ stays bounded, so the second term converges to zero (for a single asset this is verified explicitly via the $\epsilon = 1/\lambda$ expansion in the proof of Proposition 2). Therefore

$$\lim_{\lambda \rightarrow \infty} \hat{y}_t^s = (\Omega_t^s)^{-1} (\mathbb{E}_t(R^s) - r\mathbf{1}),$$

and we have proved the claim.

A.1.4 Equilibrium ODE System and Demand: CRRA Investors

We now consider the case with CRRA investors. A CRRA investor maximizes

$$\mathbb{E}_t \left[\int_t^\infty e^{-\rho(\tau-t)} \frac{\hat{c}_\tau^{1-\gamma}}{1-\gamma} d\tau \right], \quad (\text{A.39})$$

subject to the wealth dynamic (A.4). The investor's HJB equation can be written as

$$\begin{aligned} \rho V^\xi = \max_{\hat{c}_t^\xi, \hat{z}_t^\xi} \left\{ \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} + V_{\hat{w}}^\xi \mu_t^{\hat{w}, \xi} + \frac{1}{2} V_{\hat{w}\hat{w}}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{\hat{w}, \xi} + V_w^\xi \mu_t^{w, \xi} + \frac{1}{2} V_{ww}^\xi (\sigma_t^{w, \xi})^\top \sigma_t^{w, \xi} \right. \\ \left. + V_{\hat{w}w}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{w, \xi} + \lambda^\xi (V^{-\xi, +} - V^\xi) \right\}, \end{aligned} \quad (\text{A.40})$$

where $\mu_t^{\hat{w}, \xi}$, $\sigma_t^{\hat{w}, \xi}$, $\mu_t^{w, \xi}$ and $\sigma_t^{w, \xi}$ are given by (A.11)-(A.12).

We conjecture (and later verify) that the investor's value function takes the form

$$V^\xi(\hat{w}_t^\xi, w_t^\xi) = q^\xi(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma}. \quad (\text{A.41})$$

Substituting the conjectured value function into the investor's HJB equation (A.40) yields

$$\begin{aligned} \rho q^\xi(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} = \max_{\hat{c}_t^\xi, \hat{z}_t^\xi} \left\{ \frac{(\hat{c}_t^\xi)^{1-\gamma}}{1-\gamma} + q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{-\gamma} \mu_t^{\hat{w}, \xi} - \frac{\gamma}{2} q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{-\gamma-1} (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{\hat{w}, \xi} \right. \\ \left. + q^{\xi'}(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} \mu_t^{w, \xi} + \frac{1}{2} q^{\xi''}(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} (\sigma_t^{w, \xi})^\top \sigma_t^{w, \xi} + q^{\xi'}(w_t^\xi) (\hat{w}_t^\xi)^{-\gamma} (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{w, \xi} \right. \\ \left. + \lambda^\xi \left[q^{-\xi}(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)) \frac{[\hat{w}_t^\xi + \hat{w}_t^\xi \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^{1-\gamma}}{1-\gamma} - q^\xi(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} \right] \right\}. \end{aligned} \quad (\text{A.42})$$

Taking the first-order condition with respect to consumption yields

$$(\hat{c}_t^\xi)^{-\gamma} - q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{-\gamma} = 0. \quad (\text{A.43})$$

Thus,

$$\hat{c}_t^\xi = q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} \hat{w}_t^\xi. \quad (\text{A.44})$$

Let $\hat{z}_t^\xi \equiv \hat{Y}_t^\xi / \hat{w}_t^\xi$ denote the investor's optimal wealth-scaled asset holdings at time t if the economy is in state ξ . Let $z_t^\xi \equiv Y_t^\xi / w_t^\xi$ denote the investors' aggregate wealth-scaled asset holdings. Taking the first-order condition, $\hat{z}_t^\xi$ satisfies

$$\begin{aligned} 0 = & q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{1-\gamma} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) - \gamma q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{1-\gamma} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \hat{z}_t^\xi \\ & + q^{\xi'}(w_t^\xi) (\hat{w}_t^\xi)^{1-\gamma} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi \\ & + \lambda^\xi q^{-\xi}(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)) [1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^{-\gamma} (\hat{w}_t^\xi)^{1-\gamma} (P_t^{-\xi} - P_t^\xi). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} \bar{D} + \mu_{P_t}^\xi - rP_t^\xi = & \gamma(\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \hat{z}_t^\xi - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\ & - \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma}, \end{aligned} \quad (\text{A.45})$$

or equivalently

$$\begin{aligned} \hat{z}_t^\xi = & \frac{1}{\gamma} (\Pi_t^\xi)^{-1} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) + \frac{1}{\gamma} \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi z_t^\xi \\ & + \frac{\lambda^\xi q^{-\xi}(w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi))}{\gamma q^\xi(w_t^\xi) [1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma} (\Pi_t^\xi)^{-1} (P_t^{-\xi} - P_t^\xi), \end{aligned}$$

where $\Pi_t^\xi \equiv (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma)$. Since $\hat{y}_t^\xi = \text{diag}(P_t^\xi) \hat{z}_t^\xi$, then

$$\begin{aligned} \hat{y}_t^\xi = & \frac{1}{\gamma} (\Omega_t^\xi)^{-1} (\mu_{Rt}^\xi - r\mathbf{1} + \lambda^\xi J_{Rt}^\xi) + \frac{1}{\gamma} \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi y_t^\xi \\ & + \frac{\lambda^\xi}{\gamma} (\Omega_t^\xi)^{-1} \left(\frac{q^{-\xi}(w_t^\xi (1 + y_t^{\xi\top} J_{Rt}^\xi))}{q^\xi(w_t^\xi) [1 + \hat{y}_t^{\xi\top} J_{Rt}^\xi]^\gamma} - 1 \right) J_{Rt}^\xi. \end{aligned}$$

Note that $\hat{z}_t^\xi = z_t^\xi$, $\hat{c}_t^\xi = c_t^\xi$ and $\hat{w}_t^\xi = w_t^\xi$ in equilibrium, then (A.45) implies

$$\begin{aligned} \bar{D} + \mu_{P_t}^\xi - rP_t^\xi = & \left[\gamma - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi \right] (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\ & - \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma}. \end{aligned}$$

Let

$$\Gamma^\xi(w_t^\xi) \equiv \gamma - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi, \quad (\text{A.46})$$

Then, the above equation can be rewritten as

$$\begin{aligned} \bar{D} + \mu_{P_t}^\xi - rP_t^\xi = & \Gamma^\xi(w_t^\xi) (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\ & - \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma}. \end{aligned} \quad (\text{A.47})$$

Moreover, from (A.44), the investors' total consumption aggregates to $c_t^\xi = q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} w_t^\xi$.

Substituting it into the HJB equation (A.42) yields

$$\begin{aligned}
\rho q^\xi(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} &= \frac{q^\xi(w_t^\xi)^{-\frac{1-\gamma}{\gamma}} (\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} \\
&+ q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{1-\gamma} \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + \hat{z}_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right] \\
&- \frac{\gamma}{2} q^\xi(w_t^\xi) (\hat{w}_t^\xi)^{1-\gamma} \hat{z}_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \hat{z}_t^\xi \\
&+ q^{\xi'}(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} w_t^\xi \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right] \\
&+ \frac{1}{2} q^{\xi''}(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} (w_t^\xi)^2 z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\
&+ q^{\xi'}(w_t^\xi) (\hat{w}_t^\xi)^{1-\gamma} w_t^\xi \hat{z}_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\
&+ \lambda^\xi \left[q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) \frac{[1 + \hat{z}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^{1-\gamma} (\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} - q^\xi(w_t^\xi) \frac{(\hat{w}_t^\xi)^{1-\gamma}}{1-\gamma} \right].
\end{aligned}$$

Note that $\hat{w}_t^{1-\gamma}$ cancels out on both sides. Thus, we have verified the conjectured value function (A.41). Using $\hat{z}_t^\xi = z_t^\xi$, we get

$$\begin{aligned}
\rho q^\xi(w_t^\xi) &= q^\xi(w_t^\xi)^{-\frac{1-\gamma}{\gamma}} + \left[q^{\xi'}(w_t^\xi) w_t^\xi + (1-\gamma) q^\xi(w_t^\xi) \right] \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - r P_t^\xi) \right] \\
&+ \frac{1}{2} \left[q^{\xi''}(w_t^\xi) (w_t^\xi)^2 - \gamma(1-\gamma) q^\xi(w_t^\xi) + 2(1-\gamma) q^{\xi'}(w_t^\xi) w_t^\xi \right] z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\
&+ \lambda^\xi \left[q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^{1-\gamma} - q^\xi(w_t^\xi) \right].
\end{aligned} \tag{A.48}$$

Using (A.47) and dividing both sides by $q^\xi(w_t^\xi)$ and rearranging, we find

$$\begin{aligned}
q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} &+ \left[\frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi + 1 - \gamma \right] \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + \Gamma^\xi(w_t^\xi) z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \right] \\
&- \left[\frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi + 1 - \gamma \right] \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma} \\
&+ \frac{1}{2} \left[\frac{q^{\xi''}(w_t^\xi)}{q^\xi(w_t^\xi)} (w_t^\xi)^2 - \gamma(1-\gamma) + 2(1-\gamma) \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi \right] z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \\
&+ \lambda^\xi \left[\frac{q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)])}{q^\xi(w_t^\xi)} [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^{1-\gamma} - 1 \right] = \rho.
\end{aligned} \tag{A.49}$$

From (A.46),

$$\begin{aligned}\Gamma^{\xi'}(w_t^\xi) &= -\frac{q^{\xi''}(w_t^\xi)q^\xi(w_t^\xi) - q^{\xi'}(w_t^\xi)^2}{q^\xi(w_t^\xi)^2}w_t^\xi - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} \\ &= -\frac{q^{\xi''}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi + \frac{q^{\xi'}(w_t^\xi)^2}{q^\xi(w_t^\xi)^2}w_t^\xi - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{q^{\xi''}(w_t^\xi)}{q^\xi(w_t^\xi)}(w_t^\xi)^2 - \gamma(1 - \gamma) + 2(1 - \gamma)\frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi \\ &= \left[\frac{q^{\xi''}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi - \frac{q^{\xi'}(w_t^\xi)^2}{q^\xi(w_t^\xi)^2}w_t^\xi + \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}\right]w_t^\xi + \left[\frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi - \gamma\right]^2 + \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi - \gamma \\ &= -\Gamma^{\xi'}(w_t^\xi)w_t^\xi + \Gamma^\xi(w_t^\xi)^2 - \Gamma^\xi(w_t^\xi).\end{aligned}$$

Substituting into (A.49) and subtracting r from both sides, we get

$$\begin{aligned}\rho - r &= q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - r + \left[\frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi + 1 - \gamma\right]\left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}}\right] \\ &\quad - \lambda^\xi \left[\frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)}w_t^\xi + 1 - \gamma\right] \frac{q^{-\xi}(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma} \\ &\quad + \frac{1}{2}\left[-\Gamma^{\xi'}(w_t^\xi)w_t^\xi + \Gamma^\xi(w_t^\xi)^2 - \Gamma^\xi(w_t^\xi) + 2[1 - \Gamma^\xi(w_t^\xi)]\Gamma^\xi(w_t^\xi)\right]z_t^{\xi\top}(\sigma_{P_t}^\xi + \sigma)^\top(\sigma_{P_t}^\xi + \sigma)z_t^\xi \\ &\quad + \lambda^\xi \left[\frac{q^{-\xi}(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]}{q^\xi(w_t^\xi)}[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^{1-\gamma} - 1\right] \\ &= \left[q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - r\right]\Gamma^\xi(w_t^\xi) - \frac{1}{2}\left[\Gamma^{\xi'}(w_t^\xi)w_t^\xi - \Gamma^\xi(w_t^\xi) + \Gamma^\xi(w_t^\xi)^2\right]z_t^{\xi\top}(\sigma_{P_t}^\xi + \sigma)^\top(\sigma_{P_t}^\xi + \sigma)z_t^\xi \\ &\quad + \lambda^\xi \left\{\frac{[\Gamma^\xi(w_t^\xi)z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi) + 1]q^{-\xi}(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma} - 1\right\}.\end{aligned}\tag{A.50}$$

Substituting the investors' aggregate consumption c_t^ξ into the total wealth dynamic for investors, we get

$$\begin{aligned}dw_t^\xi &= w_t^\xi \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + z_t^{\xi\top}(\bar{D} + \mu_{P_t}^\xi - rP_t^\xi)\right]dt + w_t^\xi z_t^{\xi\top}(\sigma_{P_t}^\xi + \sigma)^\top dB_t \\ &\quad + w_t^\xi z_t^{\xi\top} J_t^\xi dN_t^\xi.\end{aligned}\tag{A.51}$$

The investors' total wealth w_t^ξ and the state of the economy ξ are the two state variables in the model. Thus, the risky assets' prices in state ξ must be functions of the investors' total

wealth in the state, that is, $P_t^\xi = P^\xi(w_t^\xi)$. Let

$$\tilde{\mu}_t^{w,\xi} \equiv w_t^\xi [r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi)], \quad (\text{A.52})$$

$$\tilde{\sigma}_t^{w,\xi} \equiv (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi, \quad (\text{A.53})$$

which are the drift and the diffusion of the investors' total wealth w_t^ξ after substituting in the investors' optimal consumption and wealth-scaled asset holdings. By Itô's lemma, we have

$$dP_t^\xi = \left[\tilde{\mu}_t^{w,\xi} P^{\xi'}(w_t^\xi) + \frac{1}{2} (\tilde{\sigma}_t^{w,\xi})^\top \tilde{\sigma}_t^{w,\xi} P^{\xi''}(w_t^\xi) \right] dt + P^{\xi'}(w_t^\xi) (\tilde{\sigma}_t^{w,\xi})^\top dB_t \\ + (P_t^{-\xi} - P_t^\xi) dN_t^\xi.$$

From (A.52) and (A.53), the drift of the risky assets' prices becomes

$$\begin{aligned} & \tilde{\mu}_t^{w,\xi} P^{\xi'}(w_t^\xi) + \frac{1}{2} (\tilde{\sigma}_t^{w,\xi})^\top \tilde{\sigma}_t^{w,\xi} P^{\xi''}(w_t^\xi) \\ &= w_t^\xi [r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + z_t^{\xi\top} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi)] P^{\xi'}(w_t^\xi) \\ & \quad + \frac{1}{2} z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\ &= w_t^\xi \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} + \Gamma^\xi(w_t^\xi) z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \right. \\ & \quad \left. - \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi) \\ & \quad + \frac{1}{2} z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\ &= \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ & \quad + z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right], \end{aligned}$$

where the second equality follows from (A.47). The diffusion of the risky assets' prices is

$$\tilde{\sigma}_t^{w,\xi} P^{\xi'}(w_t^\xi)^\top = (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi P^{\xi'}(w_t^\xi)^\top.$$

The risky assets' prices are conjectured to follow the Itô process (9). Thus, by matching coefficients, we get

$$J_t^\xi = P_t^{-\xi} - P_t^\xi,$$

while $\mu_{P_t}^\xi$ and $\sigma_{P_t}^\xi$ are given by

$$\begin{aligned} \mu_{P_t}^\xi &= \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]) z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ & \quad + z_t^{\xi\top} (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) z_t^\xi \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right], \end{aligned} \quad (\text{A.54})$$

and

$$\sigma_{P_t}^\xi = (\sigma_{P_t}^\xi + \sigma) z_t^\xi w_t^\xi P^{\xi'}(w_t^\xi)^\top. \quad (\text{A.55})$$

From (A.55),

$$(\sigma_{P_t}^\xi + \sigma) z_t^\xi = (\sigma_{P_t}^\xi + \sigma) z_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi + \sigma z_t^\xi.$$

Note that $P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi$ is a scalar. Thus,

$$[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi] (\sigma_{P_t}^\xi + \sigma) z_t^\xi = \sigma z_t^\xi.$$

Then

$$(\sigma_{P_t}^\xi + \sigma) z_t^\xi = \frac{\sigma z_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}. \quad (\text{A.56})$$

Substituting (A.56) into (A.55) yields

$$\sigma_{P_t}^\xi = \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}.$$

Adding σ to both sides yields

$$\sigma_{P_t}^\xi + \sigma = \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi + [1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi] \sigma}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}.$$

Let

$$\hat{\sigma}_t^\xi \equiv \sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi + [1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi] \sigma. \quad (\text{A.57})$$

Then

$$\sigma_{P_t}^\xi + \sigma = \frac{\hat{\sigma}_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi}. \quad (\text{A.58})$$

Substituting (A.56) into (A.54) yields

$$\begin{aligned} \mu_{P_t}^\xi &= \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)] z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) [1 + z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ &\quad + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right]. \end{aligned}$$

It is thus immediate that

$$\begin{aligned}
& \bar{D} + \mu_{P_t}^\xi - rP_t^\xi \\
&= \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)])z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi)w_t^\xi \\
&+ \frac{z_t^{\xi\top}\Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi)P^{\xi'}(w_t^\xi)w_t^\xi + \frac{1}{2}P^{\xi''}(w_t^\xi)(w_t^\xi)^2 \right] + \bar{D} - rP^\xi(w_t^\xi).
\end{aligned} \tag{A.59}$$

From (A.47) and using (A.56) and (A.58), we have

$$\bar{D} + \mu_{P_t}^\xi - rP_t^\xi = \frac{\Gamma^\xi(w_t^\xi)\hat{\sigma}_t^{\xi\top}\sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)])(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma}. \tag{A.60}$$

From (A.59) and (A.60),

$$\begin{aligned}
& \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)])z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi)w_t^\xi \\
&+ \frac{z_t^{\xi\top}\Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi)P^{\xi'}(w_t^\xi)w_t^\xi + \frac{1}{2}P^{\xi''}(w_t^\xi)(w_t^\xi)^2 \right] + \bar{D} - rP^\xi(w_t^\xi) \\
&= \frac{\Gamma^\xi(w_t^\xi)\hat{\sigma}_t^{\xi\top}\sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)])(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma}.
\end{aligned} \tag{A.61}$$

The equation (A.61) follows from the investors' first-order condition and Itô's lemma. Moreover, (A.60) implies that

$$\begin{aligned}
z_t^{\xi\top}(\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) &= \Gamma^\xi(w_t^\xi) \frac{z_t^{\xi\top}\Sigma z_t^\xi}{[1 - z_t^{\xi\top}P^{\xi'}(w_t^\xi)w_t^\xi]^2} \\
&- \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)])z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma}.
\end{aligned} \tag{A.62}$$

Since there is a measure one of symmetric hedgers, the aggregate risky asset holdings by hedgers at time t in state ξ are given by

$$Q_t^\xi = \left[(\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) + \lambda^\xi J_t^\xi J_t^{\xi\top} \right]^{-1} \left[\frac{\bar{D} + \mu_{P_t}^\xi - rP_t^\xi + \lambda^\xi J_t^\xi}{\alpha} - (\sigma_{P_t}^\xi + \sigma)^\top \sigma u^\xi \right]. \tag{A.63}$$

From (A.56), (A.58) and (A.59) and using the fact that $J_t^\xi = P_t^{-\xi} - P_t^\xi$,

$$\begin{aligned}
Q_t^\xi = & \frac{1}{\alpha} \left\{ \frac{\hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \lambda^\xi (P_t^{-\xi} - P_t^\xi)(P_t^{-\xi} - P_t^\xi)^\top \right\}^{-1} \left\{ \lambda^\xi (P_t^{-\xi} - P_t^\xi) \right. \\
& + \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]) z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] + \bar{D} - r P^\xi(w_t^\xi) \\
& \left. - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right\}. \tag{A.64}
\end{aligned}$$

Market clearing requires that

$$Q_t^\xi + Y_t^\xi = Q_t^\xi + z_t^\xi w_t^\xi = S. \tag{A.65}$$

Combining (A.64) and (A.65) yields

$$\begin{aligned}
& \frac{1}{\alpha} \left\{ \frac{\hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \lambda^\xi (P_t^{-\xi} - P_t^\xi)(P_t^{-\xi} - P_t^\xi)^\top \right\}^{-1} \left\{ \lambda^\xi (P_t^{-\xi} - P_t^\xi) \right. \\
& + \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^\xi[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]) z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi)[1 + z_t^{\xi\top}(P_t^{-\xi} - P_t^\xi)]^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] + \bar{D} - r P^\xi(w_t^\xi) \\
& \left. - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right\} = S - z_t^\xi w_t^\xi. \tag{A.66}
\end{aligned}$$

The equation (A.66) follows from the market clearing condition.

When the economy switches from state ξ to state $-\xi$ at time t , the investor's wealth jumps from w_t^ξ to $w_t^{-\xi}$ according to the following fixed point condition

$$w_t^{-\xi} = w_t^\xi + w_t^\xi z_t^{\xi\top} (P_t^{-\xi} - P_t^\xi). \tag{A.67}$$

Substituting (A.67) into (A.61) yields

$$\begin{aligned}
& \bar{D} - r P^\xi(w_t^\xi) + \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] \\
& = \frac{\Gamma^\xi(w_t^\xi) \hat{\sigma}_t^{\xi\top} \sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma}. \tag{A.68}
\end{aligned}$$

Furthermore, substituting (A.67) into (A.66) yields

$$\begin{aligned}
& \bar{D} - rP_t^\xi + \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi}\right)^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\
& + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} \left[\Gamma^\xi(w_t^\xi) P^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 \right] \\
& = \alpha \left\{ \frac{\hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi}{[1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi]^2} + \lambda^\xi (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top \right\} (S - z_t^\xi w_t^\xi) \\
& + \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} - \lambda^\xi (P_t^{-\xi} - P_t^\xi).
\end{aligned} \tag{A.69}$$

Combining (A.68) and (A.69) yields

$$\begin{aligned}
& \frac{\Gamma^\xi(w_t^\xi) \hat{\sigma}_t^{\xi\top} \sigma z_t^\xi}{[1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi}\right)^\gamma} \\
& = \frac{\alpha \hat{\sigma}_t^{\xi\top} \hat{\sigma}_t^\xi (S - z_t^\xi w_t^\xi)}{[1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} + \alpha \lambda^\xi (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top (S - z_t^\xi w_t^\xi) \\
& + \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi} - \lambda^\xi (P_t^{-\xi} - P_t^\xi).
\end{aligned}$$

Rearranging yields

$$\begin{aligned}
& \frac{[\Gamma^\xi(w_t^\xi) + \alpha w_t^\xi] \hat{\sigma}_t^{\xi\top} \sigma z_t^\xi - \alpha \hat{\sigma}_t^{\xi\top} \sigma (\sigma^{-1} \hat{\sigma}_t^\xi) S}{[1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi} \\
& = \lambda^\xi \left\{ \left[\frac{q^{-\xi}(w_t^{-\xi})}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi}\right)^\gamma} - 1 \right] (P_t^{-\xi} - P_t^\xi) + \alpha (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top (S - z_t^\xi w_t^\xi) \right\}.
\end{aligned} \tag{A.70}$$

Note that

$$\hat{\sigma}_t^\xi = w_t^\xi \sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top + [1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi] \sigma.$$

Thus,

$$\hat{\sigma}_t^{\xi\top} \sigma z_t^\xi = w_t^\xi P^{\xi'}(w_t^\xi) z_t^{\xi\top} \Sigma z_t^\xi + [1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi] \Sigma z_t^\xi.$$

Then, (A.68) simplifies to

$$\begin{aligned} \bar{D} - rP^\xi(w_t^\xi) + \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma} \right] P^{\xi'}(w_t^\xi) w_t^\xi \\ + \frac{z_t^{\xi\top} \Sigma z_t^\xi}{2[1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} P^{\xi''}(w_t^\xi) (w_t^\xi)^2 = \frac{\Gamma^\xi(w_t^\xi) \Sigma z_t^\xi}{1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) (P_t^{-\xi} - P_t^\xi)}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma}. \end{aligned} \quad (\text{A.71})$$

Moreover, $q^\xi(w_t^\xi)$ is the solution to the ODE (A.50). Using (A.56), we can rewrite (A.50) as

$$\begin{aligned} \frac{q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - r}{\rho - r} \Gamma^\xi(w_t^\xi) - \frac{\Gamma^{\xi'}(w_t^\xi) w_t^\xi - \Gamma^\xi(w_t^\xi) + \Gamma^\xi(w_t^\xi)^2}{2(\rho - r)} \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 - w_t^\xi P^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} \\ + \frac{\lambda^\xi}{\rho - r} \left\{ \frac{[\Gamma^\xi(w_t^\xi) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi} + 1] q^{-\xi}(w_t^{-\xi})}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma} - 1 \right\} = 1. \end{aligned} \quad (\text{A.72})$$

The investors' wealth-scaled asset holdings z_t^ξ are determined by (A.70), while prices $P^\xi(w_t^\xi)$ and hedging demand $q^\xi(w_t^\xi)$ are solutions to the ODEs (A.71) and (A.72).

Summary of the ODE system. We can define $\pi^\xi(w_t^\xi) \equiv \frac{\bar{D}}{r} - P^\xi(w_t^\xi)$. Then, $P^{\xi'}(w_t^\xi) = -\pi^{\xi'}(w_t^\xi)$ and $P^{\xi''}(w_t^\xi) = -\pi^{\xi''}(w_t^\xi)$. The ODE (A.71) becomes

$$\begin{aligned} r\pi^\xi(w_t^\xi) - \left[r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi}}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma} \right] \pi^{\xi'}(w_t^\xi) w_t^\xi - \frac{z_t^{\xi\top} \Sigma z_t^\xi \pi^{\xi''}(w_t^\xi) (w_t^\xi)^2}{2[1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} \\ = \frac{\Gamma^\xi(w_t^\xi) \Sigma z_t^\xi}{1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi} - \frac{\lambda^\xi q^{-\xi}(w_t^{-\xi}) [\pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi})]}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma}. \end{aligned}$$

The ODE (A.72) becomes

$$\begin{aligned} \frac{q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} - r}{\rho - r} \Gamma^\xi(w_t^\xi) - \frac{\Gamma^{\xi'}(w_t^\xi) w_t^\xi - \Gamma^\xi(w_t^\xi) + \Gamma^\xi(w_t^\xi)^2}{2(\rho - r)} \frac{z_t^{\xi\top} \Sigma z_t^\xi}{[1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} \\ + \frac{\lambda^\xi}{\rho - r} \left\{ \frac{[\Gamma^\xi(w_t^\xi) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi} + 1] q^{-\xi}(w_t^{-\xi})}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi} \right)^\gamma} - 1 \right\} = 1. \end{aligned}$$

Moreover, (A.70) can be written as

$$\begin{aligned} & \frac{[\Gamma^\xi(w_t^\xi) + \alpha w_t^\xi] \hat{\sigma}_t^{\xi\top} \sigma z_t^\xi - \alpha \hat{\sigma}_t^{\xi\top} \sigma (\sigma^{-1} \hat{\sigma}_t^\xi) S}{[1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi]^2} - \frac{\alpha \hat{\sigma}_t^{\xi\top} \sigma u^\xi}{1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi} \\ &= \lambda^\xi \left\{ \left[\frac{q^{-\xi}(w_t^{-\xi})}{q^\xi(w_t^\xi) \left(\frac{w_t^{-\xi}}{w_t^\xi}\right)^\gamma} - 1 \right] (P_t^{-\xi} - P_t^\xi) + \alpha (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top (S - z_t^\xi w_t^\xi) \right\}. \end{aligned}$$

Here,

$$\begin{aligned} \hat{\sigma}_t^\xi &= -w_t^\xi \sigma z_t^\xi \pi^{\xi'}(w_t^\xi)^\top + [1 + w_t^\xi \pi^{\xi'}(w_t^\xi)^\top z_t^\xi] \sigma, \\ \Gamma^\xi(w_t^\xi) &= \gamma - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} w_t^\xi. \end{aligned}$$

Moreover, the following fixed point condition must be satisfied

$$w_t^{-\xi} - w_t^\xi = w_t^\xi z_t^{\xi\top} [\pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi})].$$

Variance-covariance matrix of percentage returns. As in the case with log investors, the endogenous percentage return variance-covariance matrix is given by $\Omega_t^\xi \equiv (\sigma_{Rt}^\xi)^\top \sigma_{Rt}^\xi$, where $\sigma_{Rt}^\xi \equiv (\sigma_{Pt}^\xi + \sigma) \text{diag}(P_t^\xi)^{-1}$. Then

$$\begin{aligned} \Omega_t^\xi &= \text{diag}(P_t^\xi)^{-1} (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma) \text{diag}(P_t^\xi)^{-1} \\ &= \text{diag}(P_t^\xi)^{-1} \left[\sigma + \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right]^\top \left[\sigma + \frac{\sigma z_t^\xi P^{\xi'}(w_t^\xi)^\top w_t^\xi}{1 - P^{\xi'}(w_t^\xi)^\top z_t^\xi w_t^\xi} \right] \text{diag}(P_t^\xi)^{-1}. \end{aligned} \quad (\text{A.73})$$

The variance-covariance matrix exhibits a factor structure and depends on the endogenous state variable, the investors' total wealth w_t^ξ . □

A.2 Proof of Proposition 2

Based on the derivations for Proposition 1 and Proposition 5 and specializing the ODEs to the one-asset case, we find that investors' aggregate wealth-scaled asset holding z_t^ξ satisfies

$$\begin{aligned} & \frac{(1 + \alpha w_t^\xi) \sigma^2 z_t^\xi - \alpha \sigma^2 S}{[1 + \pi^{\xi'}(w_t^\xi) w_t^\xi z_t^\xi]^2} - \frac{\alpha \sigma^2 u^\xi}{1 + \pi^{\xi'}(w_t^\xi) w_t^\xi z_t^\xi} \\ &= \lambda^\xi \left[\left(\frac{w_t^\xi}{w_t^{-\xi}} - 1 \right) \frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi z_t^\xi} + \alpha (S - w_t^\xi z_t^\xi) \left(\frac{w_t^{-\xi} - w_t^\xi}{w_t^\xi z_t^\xi} \right)^2 \right]. \end{aligned}$$

The functions $\pi^\xi(w_t^\xi)$ and $\bar{q}^\xi(w_t^\xi)$ are solutions to the following ODEs

$$\begin{aligned} r\pi^\xi(w_t^\xi) - \left[r - \rho - \lambda^\xi \frac{w_t^{-\xi} - w_t^\xi}{w_t^{-\xi}} \right] \pi^{\xi'}(w_t^\xi) w_t^\xi - \frac{1}{2} \left[\frac{\sigma z_t^\xi}{1 + \pi^{\xi'}(w_t^\xi) w_t^\xi z_t^\xi} \right]^2 \pi^{\xi''}(w_t^\xi) (w_t^\xi)^2 \\ = \frac{\sigma^2 z_t^\xi}{1 + \pi^{\xi'}(w_t^\xi) w_t^\xi z_t^\xi} - \lambda^\xi \frac{\pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi})}{\frac{w_t^{-\xi}}{w_t^\xi}}, \end{aligned} \quad (\text{A.74})$$

and

$$\begin{aligned} \log \rho + \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right] (r - \rho) - \rho \bar{q}^\xi(w_t^\xi) \\ + \left[\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{2} \bar{q}^{\xi''}(w_t^\xi) (w_t^\xi)^2 + \frac{1}{2\rho} \right] \left[\frac{\sigma z_t^\xi}{1 + \pi^{\xi'}(w_t^\xi) w_t^\xi z_t^\xi} \right]^2 \\ + \lambda^\xi \left[\frac{1}{\rho} \log \left(\frac{w_t^{-\xi}}{w_t^\xi} \right) + \bar{q}^{-\xi}(w_t^{-\xi}) - \bar{q}^\xi(w_t^\xi) - \left(\bar{q}^{\xi'}(w_t^\xi) w_t^\xi + \frac{1}{\rho} \right) \frac{w_t^{-\xi} - w_t^\xi}{w_t^{-\xi}} \right] = 0. \end{aligned} \quad (\text{A.75})$$

Moreover, the following fixed point condition must be satisfied

$$w_t^{-\xi} - w_t^\xi = w_t^\xi z_t^\xi [\pi^\xi(w_t^\xi) - \pi^{-\xi}(w_t^{-\xi})]. \quad (\text{A.76})$$

From (A.27) and (A.28) and specializing to the one-asset case, we get

$$\sigma_{Pt}^\xi + \sigma = \frac{\sigma}{1 + \pi^{\xi'}(w_t^\xi) w_t^\xi z_t^\xi} = \frac{\sigma}{1 + \pi^{\xi'}(w_t^\xi) (S - Q_t^\xi)}.$$

The second equality follows from the market clearing condition $w_t z_t^\xi = S - Q_t^\xi$. Then

$$\sigma_{Rt}^\xi \equiv \frac{\sigma_{Pt}^\xi + \sigma}{P_t^\xi} = \frac{\sigma}{[1 + \pi^{\xi'}(w_t^\xi) (S - Q_t^\xi)] [\frac{\bar{D}}{r} - \pi^\xi(w_t^\xi)]}. \quad (\text{A.77})$$

Note that in the one-asset case,

$$Q_t^\xi = \left[(\sigma_{Pt}^\xi + \sigma)^2 + \lambda^\xi (J_t^\xi)^2 \right]^{-1} \left[\frac{\bar{D} + \mu_{Pt}^\xi - r P_t^\xi + \lambda^\xi J_t^\xi}{\alpha} - (\sigma_{Pt}^\xi + \sigma) \sigma u^\xi \right]. \quad (\text{A.78})$$

Suppose that we are standing at the after-shock state $\xi = s$. We want to analyze the asymptotic behavior at $\lambda^s = \lambda \rightarrow \infty$. Let $\epsilon \equiv 1/\lambda$. Expand the function $\pi^s(w)$ according to

$$\pi^s(w) = \pi^n(w) + \epsilon \pi_1(w) + \epsilon^2 \pi_2(w) + \dots$$

where higher-order terms have been omitted. Suppose $w_t^s = w$, then from the fixed point condition (A.76), we have

$$\begin{aligned} w_t^n &= w - (S - Q_t^n) [\pi^n(w_t^n) - \pi^s(w)] \\ &= w - (S - Q_t^n) [\pi^n(w_t^n) - \pi^n(w) - \epsilon \pi_1(w) - \epsilon^2 \pi_2(w)]. \end{aligned}$$

For small ϵ , we have

$$w_t^n = w - (S - Q_t^n) [\pi^{n'}(w)(w_t^n - w) - \epsilon \pi_1(w) - \epsilon^2 \pi_2(w)],$$

which implies

$$w_t^n = w + \frac{S - Q_t^n}{1 + (S - Q_t^n) \pi^{n'}(w)} [\pi_1(w) \epsilon + \pi_2(w) \epsilon^2].$$

When $\lambda \rightarrow \infty$, $w_t^n \rightarrow w$ and $\pi^n(w_t^n) \rightarrow \pi^s(w)$. Hence, in the limit with $\lambda \rightarrow \infty$, (A.17) implies

$$(\sigma_{Pt}^s + \sigma)^2 \hat{z}_t^s = \bar{D} + \mu_{Pt}^s - r P_t^s + \lambda J_t^s,$$

where λJ_t^s is bounded away from zero, and (A.78) becomes

$$Q_t^s = \frac{1}{\alpha \hat{w}_t^s} (S - Q_t^s) - \frac{\sigma}{\sigma_{Pt}^s + \sigma} u^s = \frac{1}{\alpha \hat{w}_t^s} (S - Q_t^s) - [1 + \pi^{s'}(w_t^s)(S - Q_t^s)] u^s.$$

Rearranging yields

$$Q_t^s = \frac{\frac{S}{\alpha \hat{w}_t^s} - [1 + \pi^{s'}(w_t^s) S] u^s}{1 + \frac{1}{\alpha \hat{w}_t^s} - \pi^{s'}(w_t^s) u^s}. \quad (\text{A.79})$$

Similarly, in the normal state,

$$Q_t^n = \frac{\frac{S}{\alpha \hat{w}_t^n} - [1 + \pi^{n'}(w_t^n) S] u^n}{1 + \frac{1}{\alpha \hat{w}_t^n} - \pi^{n'}(w_t^n) u^n}. \quad (\text{A.80})$$

As $\lambda \rightarrow \infty$, the two states share the same wealth and price function ($w_t^s = w_t^n = w$ and $\pi^s = \pi^n = \pi$, established above), so Q_t^ξ differs across states only through u^ξ . Differentiating with respect to u^ξ yields

$$\frac{\partial Q_t^\xi}{\partial u^\xi} = - \frac{1 + \frac{1}{\alpha w} + \pi'(w) S}{[1 + \frac{1}{\alpha w} - \pi'(w) u^\xi]^2}. \quad (\text{A.81})$$

Notice that from (A.79) and (A.80),

$$\left[1 + \frac{1}{\alpha w} - \pi'(w) u^\xi \right] Q_t^\xi = \frac{S}{\alpha w} - [1 + \pi'(w) S] u^\xi.$$

Hence,

$$\left[1 + \pi'(w)(S - Q_t^\xi) \right] \left[1 + \frac{1}{\alpha w} - \pi'(w) u^\xi \right] = 1 + \frac{1}{\alpha w} + u'(w) S.$$

Substituting into (A.81) yields

$$\frac{\partial Q_t^\xi}{\partial u^\xi} = \frac{1 + \pi'(w)(S - Q_t^\xi)}{1 + \frac{1}{\alpha w} - \pi'(w)u^\xi},$$

where the numerator equals $\sigma/(\sigma_{P_t}^\xi + \sigma) > 0$ while the denominator is positive. Hence, Q_t^ξ decreases in u^ξ . Consequently, $Q_t^s < Q_t^n$ for $u^s > u^n$. By (A.77), with common $\pi(w)$, $\sigma_{Rt}^\xi = \sigma/\{[1 + \pi'(w)(S - Q_t^\xi)][\frac{\bar{D}}{r} - \pi(w)]\}$ and it is immediate that $\sigma_{Rt}^s > \sigma_{Rt}^n$.

Note that $\Delta \mathbb{E}_t(R) = \mu_{Rt}^s + \lambda J_{Rt}^s - \mu_{Rt}^n$, that is

$$\Delta \mathbb{E}_t(R) = \frac{(\sigma_{P_t}^s + \sigma)^2(S - Q_t^s)}{\hat{w}_t^s(\frac{\bar{D}}{r} - \pi^s(w_t^s))} - \frac{(\sigma_{P_t}^n + \sigma)^2(S - Q_t^n)}{\hat{w}_t^n(\frac{\bar{D}}{r} - \pi^n(w_t^n))}.$$

Writing $Y^\xi \equiv S - Q_t^\xi$ and using $\sigma_{P_t}^\xi + \sigma = \sigma/(1 + \pi'(w)Y^\xi)$, $\Delta \mathbb{E}_t(R) = \frac{\sigma^2}{wP} [f(Y^s) - f(Y^n)]$ with $f(Y) \equiv Y/(1 + \pi'(w)Y)^2$. Note that $f'(Y) = (1 - \pi'(w)Y)/(1 + \pi'(w)Y)^3 > 0$, since the numerator is positive as $\pi'(w) < 0$ and the denominator is positive by the condition after (A.25) and $Y^s > Y^n$. Thus, we have $\Delta \mathbb{E}_t(R) > 0$ at $\lambda \rightarrow \infty$. Moreover, $\mathbb{E}_t(R^s) - r > 0$ since a risk-averse investor holds the risky asset only at a positive premium. From (25), $\frac{1}{(\sigma_{Rt}^s)^2} - \frac{1}{(\sigma_{Rt}^n)^2} < 0$ with $\mathbb{E}_t(R^s) - r > 0$ and $\Delta \mathbb{E}_t(R) > 0$ gives $\mathcal{M}^{\log}(w_t^n; u^n, u^s) < \mathcal{C}^{\log}(w_t^n; \delta)|_{\delta=\Delta \mathbb{E}_t(R)}$.

We now take the local limit $u^s \rightarrow u^n$ after taking $\lambda \rightarrow \infty$. Let $\Delta \equiv u^s - u^n$. From the preceding argument, as $\lambda \rightarrow \infty$, we have $w_t^{n-} \rightarrow w_t^s = w$, $\pi^s(w) \rightarrow \pi^n(w) = \pi(w)$, and $\pi^{s'}(w) \rightarrow \pi^{n'}(w) = \pi'(w)$. Thus, as $u^s \rightarrow u^n$, the relevant limiting price and price-loading objects are common across states. Let $Q(u)$ denote hedgers' asset holdings in the $\lambda \rightarrow \infty$ limit, that is

$$Q(u) = \frac{\frac{S}{\alpha w} - [1 + \pi'(w)S]u}{1 + \frac{1}{\alpha w} - \pi'(w)u}.$$

Differentiating gives

$$Q'(u) = -\frac{1 + \frac{1}{\alpha w} + \pi'(w)S}{[1 + \frac{1}{\alpha w} - \pi'(w)u]^2}.$$

Thus, by Taylor expansion around u^n , we get

$$Q^s - Q^n = Q'(u^n)\Delta + O(\Delta^2). \quad (\text{A.82})$$

Since $Y^\xi = S - Q^\xi$, then

$$Y^s - Y^n = -(Q^s - Q^n) = -Q'(u)\Delta + O(\Delta^2).$$

Next, using the limiting expression for return volatility, we get

$$\sigma_{Rt} \equiv \frac{\sigma}{[1 + \pi'(w)Y(u)]P}.$$

Thus,

$$\frac{1}{\sigma_{Rt}^2} = \frac{P^2}{\sigma^2} [1 + \pi'(w)Y(u)]^2.$$

In the numerator of (25),

$$\begin{aligned} \frac{1}{(\sigma_{Rt}^s)^2} - \frac{1}{(\sigma_{Rt}^n)^2} &= \frac{P^2}{\sigma^2} ([1 + \pi'(w)Y^s]^2 - [1 + \pi'(w)Y^n]^2) \\ &= \frac{2P^2}{\sigma^2} [1 + \pi'(w)Y^n] \pi'(w)(Y^s - Y^n) + O((Y^s - Y^n)^2). \end{aligned}$$

The last equality follows from Taylor expansion around Y^n . Since (A.82), then

$$\frac{1}{(\sigma_{Rt}^s)^2} - \frac{1}{(\sigma_{Rt}^n)^2} = -\frac{2P^2}{\sigma^2} [1 + \pi'(w)Y^n] \pi'(w) Q'(u) \Delta + O(\Delta^2). \quad (\text{A.83})$$

That is, the numerator is first order in Δ .

Now consider the denominator. In the same limiting economy,

$$\begin{aligned} \Delta \mathbb{E}_t(R) &= \frac{\sigma^2 Y^s}{wP[1 + \pi'(w)Y^s]^2} - \frac{\sigma^2 Y^n}{wP[1 + \pi'(w)Y^n]^2} \\ &= \frac{\sigma^2}{wP} \frac{1 - \pi'(w)Y^n}{[1 + \pi'(w)Y^n]^3} (Y^s - Y^n) + O((Y^s - Y^n)^2) \\ &= -\frac{\sigma^2}{wP} \frac{1 - \pi'(w)Y^n}{[1 + \pi'(w)Y^n]^3} Q'(u) \Delta + O(\Delta^2). \end{aligned} \quad (\text{A.84})$$

Hence, the denominator is also first order in Δ .

Combining (A.83) and (A.84),

$$\lim_{u^s \rightarrow u^n} \frac{\frac{1}{(\sigma_{Rt}^s)^2} - \frac{1}{(\sigma_{Rt}^n)^2}}{\Delta \mathbb{E}_t(R)} = \frac{2wP^3}{\sigma^4} \frac{[1 + \pi'(w)Y^n]^4}{1 - \pi'(w)Y^n} \pi'(w) < 0.$$

□

A.3 Proof of Lemma 1 and Proposition 3

We focus on the following limit, which proves the first part of Proposition 3; Lemma 1 and the second part of the Proposition are obtained as intermediate steps along the way:

$$\begin{aligned} \lim_{\Delta v \rightarrow 0} \frac{1}{\gamma} \frac{\left[\left((\Omega_t^s)^{-1} - (\Omega_t^n)^{-1} \right) (\mathbb{E}_t(R^s) - r\mathbf{1}) \right]_1}{[\Delta \mathbb{E}_t(R)]_1} &= \\ &= -\frac{P_t^2}{\gamma} \left(I(s+v) F(w_t) + \frac{r}{D} 2\pi(w_t) \left(1 - \sigma^2 I(s+v)^2 F(w_t) \right) \right) (s+v) < 0 \end{aligned}$$

where $\frac{1}{\sigma^2 I(s+v)^2} > F(w_t) > 0$ and $\bar{F}(w_t) > 0$ are scalar functions defined by ODEs, and P_t, w_t and $\pi(\cdot)$ are the limits of P_t^ξ, w_t^ξ and $\pi^\xi(\cdot)$ for both states $\xi = s, n$. Here s, v and σ^2 are the scalars of the blue/red example of Section 4.3, where $S = s\mathbf{1}$ and $\Sigma = \sigma^2 \mathbf{I}$; the next Lemma, in contrast, is stated for the general multi-asset economy with supply vector S .

We start with the following Lemma which is based on results in Kondor-Vayanos.

Lemma A.1. *When $\lambda^s, \lambda^n \rightarrow 0$, the price is*

$$P_t^\xi = \frac{\bar{D}}{r} - \pi^\xi(w_t^\xi) \Sigma (S + u^\xi),$$

the expected return has the form of

$$\mathbb{E}_t(R^\xi) - r\mathbf{1} = \text{diag}(P_t^\xi)^{-1} (\bar{D} + \mu_{P_t}^\xi - rP_t^\xi) = \text{diag}(P_t^\xi)^{-1} B^\xi(w_t^\xi) \Sigma (S + u^\xi),$$

the investors' equilibrium position has the form of

$$Y_t^\xi = G^\xi(w_t^\xi) (S + u^\xi),$$

and the inverse of the return covariance matrix has the form of

$$(\Omega_t^\xi)^{-1} = \text{diag}(P_t^\xi) \Sigma^{-1} \text{diag}(P_t^\xi) - F^\xi(w_t^\xi) \text{diag}(P_t^\xi) (S + u^\xi) (S + u^\xi)^\top \text{diag}(P_t^\xi)$$

where $\pi^\xi(\cdot), F^\xi(\cdot), B^\xi(\cdot)$ and $G^\xi(\cdot)$ are scalar functions depending on the vectors u^ξ and S only through the terms $(S + u^\xi)^\top \Sigma (S + u^\xi)$ and $(S + u^\xi)^\top \Sigma S$.

Proof. With $\lambda^s, \lambda^n \rightarrow 0$, our model is equivalent to the positive supply, short-term hedgers' version of Kondor and Vayanos (2019, Section 6). Expected return is given by Kondor and Vayanos (2019, (6.1) in Proposition 6.2) which is, with our notation,

$$\bar{D} + \mu_{P_t}^\xi - rP_t^\xi = B^\xi(w_t^\xi) \Sigma (S + u^\xi)$$

where

$$B^\xi(w_t^\xi) = \frac{\alpha \left(\frac{1}{w_t^\xi} \right)}{\alpha + \frac{1}{w_t^\xi} + \alpha \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S} \times \\ \times \left[1 - (S + u^\xi)^\top \Sigma (S + u^\xi) \frac{\alpha \pi^{\xi'}(w_t^\xi)}{\alpha + \frac{1}{w_t^\xi} + \alpha \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S} \right]$$

and $\pi^\xi(w_t^\xi)$ solves

$$\begin{aligned} & \frac{\alpha^2 (S + u^\xi)^\top \Sigma (S + u^\xi)}{2 \left[\alpha + \frac{1}{w_t^\xi} + \alpha \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S \right]^2} \left(-\pi^{\xi''}(w_t^\xi) \right) - \left(r - q^\xi(w_t^\xi)^{-\frac{1}{\gamma}} \right) \pi^{\xi'}(w_t^\xi) w_t^\xi + r \pi^\xi(w_t^\xi) \\ &= \frac{\alpha \left(\frac{\gamma}{w_t^\xi} - \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} \right)}{\alpha + \frac{1}{w_t^\xi} + \alpha \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S}, \end{aligned} \quad (\text{A.85})$$

with boundary conditions

$$\pi^\xi(0) = \frac{\alpha}{r}, \quad \lim_{w_t \rightarrow \infty} \pi^\xi(w_t) = 0.$$

We import several results from Kondor and Vayanos (2019); we first record the correspondence with our notation. Their asset-price function $S(w_t) = \frac{\bar{D}}{r} + g(w_t) \Sigma(s + u)$ (their equation (42), written in their notation) corresponds to our $P_t^\xi = \frac{\bar{D}}{r} - \pi^\xi(w_t^\xi) \Sigma(S + u^\xi)$: their $s + u$ is our $S + u^\xi$ and π^ξ corresponds to $-g$. Their coefficient of dynamic risk aversion $A(w_t) = \frac{\gamma}{w_t} - \frac{q'(w_t)}{q(w_t)}$ (their equation (19)) collects the static risk-aversion and intertemporal-hedging terms that, in our parameterization, enter through $\frac{1}{w_t^\xi}$ and $\pi^{\xi'}(w_t^\xi)$. Their price-impact diffusion $\sigma_{St} = \frac{\alpha}{\alpha + A(w_t) - F'(w_t)} \sigma u S'(w_t)^\top$ (their equation (IA94)) corresponds to our $\sigma_{P_t}^\xi$, and their arbitrageur position $y_t = \frac{\alpha}{\alpha + A(w_t) - F'(w_t)} u$ (their equation (23), with multi-asset counterpart (IA148)) corresponds to our Y_t^ξ . The function $F(w_t)$ appearing in these expressions is their hedgers' value-function component, unrelated to our F^ξ above; for the short-lived hedgers it vanishes, so $F'(w_t) = 0$.

Kondor and Vayanos (2019, Theorem 3) implies that $\pi^\xi(\cdot)$ is monotonically decreasing. For Π_t^ξ , the proof of Proposition 7 (their equation (IA94)) in Kondor and Vayanos (2019) gives, in our notation,

$$\sigma_{P_t}^\xi = -\frac{\alpha}{\alpha + \frac{1}{w_t^\xi}} \pi^{\xi'}(w_t^\xi) b_t (S + u^\xi)^\top \Sigma, \quad b_t = \frac{\sigma(S + u^\xi)}{1 + \frac{\alpha}{\alpha + \frac{1}{w_t^\xi}} \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S},$$

so that, substituting b_t and collecting terms,

$$\sigma_{P_t}^\xi = \hat{F}^\xi(w_t^\xi) \sigma(S + u^\xi) (S + u^\xi)^\top \Sigma$$

with

$$\hat{F}^\xi(w_t^\xi) \equiv \frac{-\alpha \pi^{\xi'}(w_t^\xi)}{\alpha + \frac{1}{w_t^\xi} + \alpha \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S}.$$

Note that the same denominator, $\alpha + \frac{1}{w_t^\xi} + \alpha \pi^{\xi'}(w_t^\xi) (S + u^\xi)^\top \Sigma S$, the feedback factor of

Kondor and Vayanos (2019), is common to B^ξ , $\hat{F}^\xi$ and G^ξ below, and serves as a consistency check across these expressions.

Then, writing $\sigma_{P_t}^\xi + \sigma = \sigma \left(\mathbf{I} + \hat{F}^\xi \left(w_t^\xi \right) \left(S + u^\xi \right) \left(S + u^\xi \right)^\top \Sigma \right)$ and using $\sigma^\top \sigma = \Sigma$,

$$\begin{aligned} \Pi_t^\xi &= \left(\sigma_{P_t}^\xi + \sigma \right)^\top \left(\sigma_{P_t}^\xi + \sigma \right) = \\ &= \left(\mathbf{I} + \hat{F}^\xi \left(w_t^\xi \right) \Sigma \left(S + u^\xi \right) \left(S + u^\xi \right)^\top \right) \Sigma \left(\mathbf{I} + \hat{F}^\xi \left(w_t^\xi \right) \left(S + u^\xi \right) \left(S + u^\xi \right)^\top \Sigma \right) \\ &= \hat{F}^\xi \left(w_t^\xi \right) \left(\hat{F}^\xi \left(w_t^\xi \right) \left(S + u^\xi \right)^\top \Sigma \left(S + u^\xi \right) + 2 \right) \Sigma \left(S + u^\xi \right) \left(S + u^\xi \right)^\top \Sigma + \Sigma, \end{aligned}$$

where the quadratic term collects the scalar $\left(S + u^\xi \right)^\top \Sigma \left(S + u^\xi \right)$. Defining

$$\begin{aligned} \bar{F}^\xi \left(w_t^\xi \right) &= \hat{F}^\xi \left(w_t^\xi \right) \left(\hat{F}^\xi \left(w_t^\xi \right) \left(S + u^\xi \right)^\top \Sigma \left(S + u^\xi \right) + 2 \right) = \\ &= \frac{-\alpha \pi^{\xi'} \left(w_t^\xi \right)}{\frac{1}{w_t^\xi} + \alpha \left(1 + \pi^{\xi'} \left(w_t^\xi \right) \left(S + u^\xi \right)^\top \Sigma S \right)} \left(\frac{-\alpha \pi^{\xi'} \left(w_t^\xi \right)}{\frac{1}{w_t^\xi} + \alpha \left(1 + \pi^{\xi'} \left(w_t^\xi \right) \left(S + u^\xi \right)^\top \Sigma S \right)} \left(S + u^\xi \right)^\top \Sigma \left(S + u^\xi \right) + 2 \right) \end{aligned}$$

we have simply

$$\Pi_t^\xi = \bar{F}^\xi \left(w_t^\xi \right) \Sigma \left(S + u^\xi \right) \left(S + u^\xi \right)^\top \Sigma + \Sigma$$

where $\bar{F}^\xi \left(w_t^\xi \right)$ is a scalar. For the inverse, note that by the Sherman-Morrison formula

$$\left(\Sigma + v v^\top \right)^{-1} = \Sigma^{-1} - \frac{\Sigma^{-1} v v^\top \Sigma^{-1}}{1 + v^\top \Sigma^{-1} v}.$$

Picking

$$v = \sqrt{\bar{F}^\xi \left(w_t^\xi \right) \Sigma} \left(S + u^\xi \right)$$

gives $\left(\Pi_t^\xi \right)^{-1} = \left(\Sigma + v v^\top \right)^{-1}$ and

$$\begin{aligned} \left(\Pi_t^\xi \right)^{-1} &= \Sigma^{-1} - \frac{\Sigma^{-1} v v^\top \Sigma^{-1}}{1 + v^\top \Sigma^{-1} v} \\ &= \Sigma^{-1} - \frac{\bar{F}^\xi \left(w_t^\xi \right)}{1 + \bar{F}^\xi \left(w_t^\xi \right) \left(S + u^\xi \right)^\top \Sigma \left(S + u^\xi \right)} \left(S + u^\xi \right) \left(S + u^\xi \right)^\top. \end{aligned}$$

Introducing the definition

$$F^\xi \left(w_t^\xi \right) \equiv \frac{\bar{F}^\xi \left(w_t^\xi \right)}{1 + \bar{F}^\xi \left(w_t^\xi \right) \left(S + u^\xi \right)^\top \Sigma \left(S + u^\xi \right)}$$

and using the definition of Ω_t^ξ gives that the inverse of the return covariance matrix has the form of

$$\left(\Omega_t^\xi\right)^{-1} = \text{diag}\left(P_t^\xi\right) \Sigma^{-1} \text{diag}\left(P_t^\xi\right) - F^\xi\left(w_t^\xi\right) \text{diag}\left(P_t^\xi\right) \left(S + u^\xi\right) \left(S + u^\xi\right)^\top \text{diag}\left(P_t^\xi\right).$$

Finally, (Kondor and Vayanos, 2019, Equation (IA148)) gives equilibrium positions with the choice of $F'(w_t) = 0$ as

$$Y_t^\xi = \frac{\alpha}{\alpha + \frac{1}{w_t^\xi} + \alpha\pi^{\xi'}} \left(w_t^\xi\right) \left(S + u^\xi\right)^\top \Sigma S \left(\sigma_{P_t}^\xi + \sigma\right)^{-1} \sigma \left(S + u^\xi\right).$$

Left-multiplying by $\sigma^\top \left(\sigma_{P_t}^\xi + \sigma\right)$ and noting that $\sigma^\top \sigma_{P_t}^\xi = \hat{F}^\xi\left(w_t^\xi\right) \Sigma \left(S + u^\xi\right) \left(S + u^\xi\right)^\top \Sigma$,

$$\begin{aligned} \left(\hat{F}^\xi\left(w_t^\xi\right) \Sigma \left(S + u^\xi\right) \left(S + u^\xi\right)^\top \Sigma + \Sigma\right) Y_t^\xi &= \\ &= \frac{\alpha}{\alpha + \frac{1}{w_t^\xi} + \alpha\pi^{\xi'}} \frac{\Sigma \left(S + u^\xi\right)}{\left(w_t^\xi\right) \left(S + u^\xi\right)^\top \Sigma S} \Sigma \left(S + u^\xi\right). \end{aligned}$$

The matrix on the left is invertible and maps $S + u^\xi$ to the vector $\Sigma \left(S + u^\xi\right)$ scaled by the constant $1 + \hat{F}^\xi\left(w_t^\xi\right) \left(S + u^\xi\right)^\top \Sigma \left(S + u^\xi\right)$, so

$$Y_t^\xi = G^\xi\left(w_t^\xi\right) \left(S + u^\xi\right)$$

with

$$G^\xi\left(w_t^\xi\right) = \frac{\alpha}{\alpha + \frac{1}{w_t^\xi} + \alpha\pi^{\xi'}} \frac{1}{\left(w_t^\xi\right) \left(S + u^\xi\right)^\top \Sigma S 1 + \hat{F}^\xi\left(w_t^\xi\right) \left(S + u^\xi\right)^\top \Sigma \left(S + u^\xi\right)},$$

which is the stated form. $\square$

We now return to the proof of Proposition 3. First note that in our example u^s is a permutation of u^n , so $(S + u^s)^\top \Sigma (S + u^s) = (S + u^n)^\top \Sigma (S + u^n)$ and $(S + u^s)^\top \Sigma S = (S + u^n)^\top \Sigma S$. Hence the Lemma implies that $\pi^\xi(\cdot)$, $F^\xi(\cdot)$, $B^\xi(\cdot)$ and $G^\xi(\cdot)$ are the same functions in the states $\xi = n, s$, and in the rest of the proof we omit the superscript ξ on them. Moreover, as $\Sigma = \sigma^2 \mathbf{I}$ and $S = s \mathbf{1}$, the expressions below are scalar, with s denoting the per-asset supply.

We are interested in

$$\left[\left((\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right) (\mathbb{E}_t(R^s) - r \mathbf{1})\right]_1,$$

a linear combination of the terms $\left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right]_{1j}$ with weights $[\mathbb{E}_t(R^s) - r \mathbf{1}]_j$. So con-

sider first

$$\begin{aligned} \left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1} \right]_{11} &= \left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s + v + \Delta v) \right)^2 \left(\frac{1}{\sigma^2} - F(w_t^s) (s + v + \Delta v)^2 \right) \\ &\quad - \left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s + v) \right)^2 \left(\frac{1}{\sigma^2} - F(w_t^n) (s + v)^2 \right). \end{aligned}$$

Hence, to a first order we have

$$\begin{aligned} \left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1} \right]_{11} &\approx \frac{\partial \left[\left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s + v) \right)^2 \left(\frac{1}{\sigma^2} - F(w_t) (s + v)^2 \right) \right]}{\partial w_t} (w_t^s - w_t^n) + \\ &\quad + \frac{\partial \left[\left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s + v) \right)^2 \left(\frac{1}{\sigma^2} - F(w_t) (s + v)^2 \right) \right]}{\partial v} \Delta v. \end{aligned}$$

We first show that the first term is of order $(\Delta v)^2$. The jump in wealth is the revaluation of the position $\hat{Y}_t^n$ held when the state switches, so

$$\begin{aligned} w_t^s - w_t^n &= \hat{Y}_t^{n\top} (P_t^s - P_t^n) = \hat{Y}_t^{n\top} \Sigma ((-\pi(w_t^s))(S + u^s) - (-\pi(w_t^n))(S + u^n)) = \\ &= G(w_t^n) (-\pi(w_t^s))(S + u^n)^\top \Sigma (S + u^s) - G(w_t^n) (-\pi(w_t^n))(S + u^n)^\top \Sigma (S + u^n) = \\ &= -G(w_t^n) (-\pi(w_t^s)) \sigma^2 (\Delta v)^2 + G(w_t^n) ((-\pi(w_t^s)) - (-\pi(w_t^n)))(S + u^n)^\top \Sigma (S + u^n) \end{aligned}$$

where the last equality uses that $u^s - u^n$ has a single entry Δv (asset 1) and a single entry $-\Delta v$ (asset I), so that $(S + u^n)^\top \Sigma (u^s - u^n) = \sigma^2 [(s + v) - (s + v + \Delta v)] \Delta v = -\sigma^2 (\Delta v)^2$. By the mean value theorem there exists $\hat{w}_t \in [w_t^n, w_t^s]$ such that

$$\pi(w_t^s) - \pi(w_t^n) = \pi'(\hat{w}_t) (w_t^s - w_t^n),$$

and solving for $w_t^s - w_t^n$ gives

$$w_t^s - w_t^n = - \frac{G(w_t^n) (-\pi(w_t^s))}{1 + G(w_t^n) (S + u^n)^\top \Sigma (S + u^n) \pi'(\hat{w}_t)} \sigma^2 (\Delta v)^2.$$

This proves Lemma 1, and it implies that in the first-order calculations below we can replace w_t^n and w_t^s with their common limit w_t . Hence,

$$\begin{aligned} \left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1} \right]_{11} &\approx \frac{\partial \left[\left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s + v) \right)^2 \left(\frac{1}{\sigma^2} - F(w_t) (s + v)^2 \right) \right]}{\partial v} \Delta v = \\ &= -2P_t (F(w_t) (s + v) P_t - \pi(w_t) (F(w_t) (s + v) \sigma^2 (s + v) - 1)) \Delta v. \end{aligned}$$

To sign this expression note that at the $\Delta v \rightarrow 0$ limit $(S + u^\xi)^\top \Sigma (S + u^\xi) = \sigma^2 I (s + v)^2$,

so that, using $\bar{F}(w_t) > 0$,

$$1 - \sigma^2 I(s+v)^2 F(w_t) = 1 - \frac{\sigma^2 I(s+v)^2 \bar{F}(w_t)}{1 + \bar{F}(w_t) \sigma^2 I(s+v)^2} = \frac{1}{1 + \bar{F}(w_t) \sigma^2 I(s+v)^2} \in (0, 1)$$

and, a fortiori, $F(w_t) \sigma^2 (s+v)^2 < 1$; this also establishes the bounds on $F(w_t)$ claimed in the statement. As $\pi(w_t), P_t, F(w_t) > 0$, it follows that $\left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right]_{11}$ is always negative.

Now consider $\left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right]_{1j}$ for $1 < j \leq I_1$. Just as before, the difference in w_t^ξ across the states contributes only at order $(\Delta v)^2$, so evaluating both states at the common w_t ,

$$\begin{aligned} \left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right]_{1j} &= \\ &= -F(w_t) \left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s+v)\right) \left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s+v+\Delta v)\right) (s+v)(s+v+\Delta v) + \\ &\quad + F(w_t) \left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s+v)\right)^2 (s+v)^2 \end{aligned}$$

and, to a first order,

$$\begin{aligned} \left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right]_{1j} &\approx \frac{\partial \left(-F(w_t) \left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s+v)\right) \left(\frac{\bar{D}}{r} - \pi(w_t) \sigma^2 (s+v+\Delta v)\right) (s+v)(s+v+\Delta v)\right)}{\partial \Delta v} \Big|_{\Delta v=0} \Delta v = \\ &= -F(w_t) [P_t^s]_1 (s+v) ([P_t^s]_1 - \pi(w_t) \sigma^2 (s+v)) \Delta v. \end{aligned}$$

For $I > j > I_1$ the same limit obtains: asset j 's factor is now the $(s+v+\Delta v)$ -based one in both states, which does not matter in the limit, while asset 1's factor again differs across the states by the same Δv shift as above. (It is also easy to see that $\left[(\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right]_{1I} = 0$, as the roles of the first and the last asset before and after the shock are symmetric.)

Then, using $\mathbb{E}_t(R^s) - r\mathbf{1} = \text{diag}(P_t^s)^{-1} B(w_t^s) \Sigma(S+u^s)$, in which every entry tends to $B(w_t) \sigma^2 (s+v) / P_t$, we obtain, to a first order,

$$\begin{aligned} \left[\left((\Omega_t^s)^{-1} - (\Omega_t^n)^{-1}\right) (\mathbb{E}_t(R^s) - r\mathbf{1})\right]_1 &= \\ &= -2 \left(F(w_t) (s+v) P_t - \pi(w_t) (F(w_t) (s+v) \sigma^2 (s+v) - 1)\right) B(w_t) \sigma^2 (s+v) \Delta v - \\ &\quad (I-2) F(w_t) (s+v) ([P_t^s]_1 - \pi(w_t) \sigma^2 (s+v)) B(w_t) \sigma^2 (s+v) \Delta v = \\ &= \left(-I(s+v) F(w_t) \frac{\bar{D}}{r} - 2\pi(w_t) \left(1 - \sigma^2 I(s+v)^2 F(w_t)\right)\right) B(w_t) \sigma^2 (s+v) \Delta v. \end{aligned}$$

We scale by the change in expected returns. As $[\mathbb{E}_t(R^\xi) - r\mathbf{1}]_1 = B(w_t^\xi) \sigma^2 \frac{s+[u^\xi]_1}{[P_t^\xi]_1}$ and $\frac{\partial}{\partial v} \frac{s+v}{P_t} = \frac{P_t + \pi(w_t) \sigma^2 (s+v)}{P_t^2} = \frac{\bar{D}}{r} \frac{1}{P_t^2}$, we have, to a first order, $[\Delta \mathbb{E}_t(R)]_1 = B(w_t) \frac{\bar{D}}{r} \frac{\sigma^2}{P_t^2} \Delta v$.

Dividing and letting $\Delta v \rightarrow 0$ gives

$$\begin{aligned} \lim_{\Delta v \rightarrow 0} \frac{\left[\left((\Omega_t^s)^{-1} - (\Omega_t^n)^{-1} \right) (\mathbb{E}_t(R^s) - r\mathbf{1}) \right]_1}{[\Delta \mathbb{E}_t(R)]_1} &= \\ &= -P_t^2 \left(I(s+v) F(w_t) + \frac{r}{\bar{D}} 2\pi(w_t) \left(1 - \sigma^2 I(s+v)^2 F(w_t) \right) \right) (s+v) \end{aligned}$$

which is negative as $F(w_t), \pi(w_t) > 0$ and by the bound established above. Dividing by γ gives the first part of the Proposition.

Finally, we turn to the second part of Proposition 3. As $\Omega_t^\xi = \text{diag}(P_t^\xi)^{-1} \Pi_t^\xi \text{diag}(P_t^\xi)^{-1}$ and $\Pi_t^\xi = \bar{F}(w_t^\xi) \Sigma (S + u^\xi) (S + u^\xi)^\top \Sigma + \Sigma$, we have

$$[\Omega_t^\xi]_{1i} = \bar{F}(w_t^\xi) \sigma^4 \frac{(s + [u^\xi]_1) (s + [u^\xi]_i)}{[P_t^\xi]_1 [P_t^\xi]_i} + \frac{\sigma^2}{([P_t^\xi]_1)^2} \delta_{i1}.$$

In our example, therefore,

$$[\Omega_t^s]_{1i} = \bar{F}(w_t^s) \sigma^4 \left\{ \begin{array}{ll} \frac{(s+v+\Delta v)^2}{\left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s+v+\Delta v) \right)^2} & \text{for } i = 1 \\ \frac{(s+v+\Delta v)(s+v)}{\left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s+v+\Delta v) \right) \left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s+v) \right)} & \text{for } I_1 \geq i > 1 \\ \frac{(s+v+\Delta v)^2}{\left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s+v+\Delta v) \right)^2} & \text{for } I > i > I_1 \\ \frac{(s+v+\Delta v)(s+v)}{\left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s+v+\Delta v) \right) \left(\frac{\bar{D}}{r} - \pi(w_t^s) \sigma^2 (s+v) \right)} & \text{for } i = I \end{array} \right\} + \frac{\sigma^2}{([P_t^s]_1)^2} \delta_{i1}$$

while

$$[\Omega_t^n]_{1i} = \bar{F}(w_t^n) \sigma^4 \left\{ \begin{array}{ll} \frac{(s+v)^2}{\left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s+v) \right)^2} & \text{for } i = 1 \\ \frac{(s+v)^2}{\left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s+v) \right)^2} & \text{for } I_1 \geq i > 1 \\ \frac{(s+v+\Delta v)(s+v)}{\left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s+v+\Delta v) \right) \left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s+v) \right)} & \text{for } I > i > I_1 \\ \frac{(s+v+\Delta v)(s+v)}{\left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s+v+\Delta v) \right) \left(\frac{\bar{D}}{r} - \pi(w_t^n) \sigma^2 (s+v) \right)} & \text{for } i = I \end{array} \right\} + \frac{\sigma^2}{([P_t^n]_1)^2} \delta_{i1}.$$

Therefore, taking the first-order Taylor expansion around $\Delta v = 0$ and using again that $w_t^s - w_t^n$ is of order $(\Delta v)^2$ (so that $\bar{F}$ and π are evaluated at the common limit w_t), we have

$$\lim_{\Delta v \rightarrow 0} \frac{[\Omega_t^s - \Omega_t^n]_{1i}}{\Delta v} = \sigma^4 \left\{ \begin{array}{ll} \frac{2}{(P_t)^3} \left[\bar{F}(w_t) (s+v) \frac{\bar{D}}{r} + \pi(w_t) \right] & \text{for } i = 1 \\ \bar{F}(w_t) \frac{\bar{D}}{r} \frac{s+v}{(P_t)^3} & \text{for } I > i > 1 \\ 0 & \text{for } i = I \end{array} \right\}$$

where both nonzero limits are positive as $\bar{F}(w_t), \pi(w_t) > 0$, which is the second part of the Proposition.

□

A.4 Proof of Proposition 4

We work in the $\lambda \rightarrow 0$ limit of the blue/red economy with $\Sigma = \sigma^2 \mathbf{I}$. By Lemma A.1, the scalar functions π, B, G depend on the state only through aggregate wealth, because swapping the labels of assets 1 and I leaves $(s + u^\xi)^\top \Sigma (s + u^\xi)$ and $(s + u^\xi)^\top \Sigma s$ unchanged. Consequently the price, position, and expected return of any asset k are *state-invariant* functions of (w_t^ξ, u_k) alone; write

$$[y_t^\xi]_k = a(w_t^\xi, u_k), \quad [\mathbb{E}_t(R^\xi)]_k = e(w_t^\xi, u_k).$$

Take $i = 1$ and $j = I$. Asset 1 is blue ($u_1 = v$) in state n and red ($u_1 = v + \Delta v$) in state s , while asset I is red ($u_I = v + \Delta v$) in state n . By Lemma 1, $w_t^s = w_t^n + \Delta w$ with $\Delta w = O((\Delta v)^2)$. A first-order Taylor expansion in wealth then gives

$$\begin{aligned} [y_t^s - y_t^n]_1 &= a(w_t^s, v + \Delta v) - a(w_t^n, v) = [a(w_t^n, v + \Delta v) - a(w_t^n, v)] + O((\Delta v)^2) \\ &= -([y_t^n]_1 - [y_t^n]_I) + O((\Delta v)^2), \end{aligned}$$

where we used $a(w_t^n, v) = [y_t^n]_1$, $a(w_t^n, v + \Delta v) = [y_t^n]_I$, and the second-order smallness of Δw . The identical expansion of e yields

$$[\Delta \mathbb{E}_t(R)]_1 = -([\mathbb{E}_t(R^n)]_1 - [\mathbb{E}_t(R^n)]_I) + O((\Delta v)^2).$$

The leading differences $[y_t^n]_1 - [y_t^n]_I$ and $[\mathbb{E}_t(R^n)]_1 - [\mathbb{E}_t(R^n)]_I$ are of order Δv , while the wealth corrections are of order $(\Delta v)^2$. Hence

$$\mathcal{M}^{\log}(w_t^n; u^n, u^s) = \frac{[y_t^s - y_t^n]_1}{[\Delta \mathbb{E}_t(R)]_1} = \frac{[y_t^n]_1 - [y_t^n]_I + O((\Delta v)^2)}{[\mathbb{E}_t(R^n)]_1 - [\mathbb{E}_t(R^n)]_I + O((\Delta v)^2)} = \mathcal{M}^{\text{cross-section}}(w_t^n; u^n)(1 + O(\Delta v)),$$

where the two minus signs cancel between numerator and denominator. Letting $\Delta v \rightarrow 0$ proves $\lim_{\Delta v \rightarrow 0} \mathcal{M}^{\log}(w_t^n; u^n, u^s) = \lim_{\Delta v \rightarrow 0} \mathcal{M}^{\text{cross-section}}(w_t^n; u^n)$. □

A.5 Proof of Proposition 6

For an investor facing the hypothetical dividend process (35), we conjecture the value function, substitute it into the HJB equation, and take first-order conditions in consumption and portfolio to obtain the wealth-scaled holdings $\hat{z}_t^h$ (with the log case first, then CRRA). Comparing these to the equilibrium holdings $\hat{z}_t^n$ and substituting into (36) gives the dynamic-slope decompositions (38) and (40).

Suppose that an investor is offered the opportunity to buy any asset with a δ additional instantaneous expected return. The hypothetical dividend process is

$$dD_t^h = \bar{D}dt + \sigma^\top dB_t + (\mathbf{I} - \text{diag}(\chi)) \text{diag}(P_t^n) \delta dt + \text{diag}(\chi) \text{diag}(P_t^n) \frac{\delta}{\lambda} dN_t^h.$$

Let $\epsilon \equiv \text{diag}(P_t^n)\delta$. Then

$$dD_t^h = \bar{D}dt + \sigma^\top dB_t + (\mathbf{I} - \text{diag}(\chi))\epsilon dt + \text{diag}(\chi)\frac{\epsilon}{\lambda}dN_t^h.$$

Given the investor's hypothetical dividend process, let $\hat{z}_t^h \equiv \hat{Y}_t^h/\hat{w}_t^h$ denote the individual investor's wealth-scaled asset holdings and let $\hat{c}_t^h$ denote the investor's consumption. The individual investor's wealth dynamic is

$$\begin{aligned} d\hat{w}_t^h &= r\hat{w}_t^h dt + \hat{w}_t^h \hat{z}_t^{h\top} (dD_t^h + dP_t^n - rP_t^n dt) - \hat{c}_t^h dt \\ &= \left[r\hat{w}_t^h - \hat{c}_t^h + \hat{w}_t^h \hat{z}_t^{h\top} (\bar{D} + \mu_{P_t}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon) \right] dt + \hat{w}_t^h \hat{z}_t^{h\top} (\sigma_{P_t}^n + \sigma)^\top dB_t \\ &\quad + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon dN_t^h. \end{aligned}$$

The corresponding aggregate wealth of investors is

$$dw_t^n = \left[rw_t^n - c_t^n + w_t^n z_t^{n\top} (\bar{D} + \mu_{P_t}^n - rP_t^n) \right] dt + w_t^n z_t^{n\top} (\sigma_{P_t}^n + \sigma)^\top dB_t.$$

A.5.1 Log Investors

The individual investor's HJB equation is

$$\begin{aligned} \rho V^h = \max_{\hat{c}_t^h, \hat{z}_t^h} \Big\{ & \log \hat{c}_t^h + V_{\hat{w}^h}^h \mu_t^{\hat{w}^h} + \frac{1}{2} V_{\hat{w}^h \hat{w}^h}^h (\sigma_t^{\hat{w}^h})^\top \sigma_t^{\hat{w}^h} + V_w^h \mu_t^{w,n} \\ & + \frac{1}{2} V_{ww}^h (\sigma_t^{w,n})^\top \sigma_t^{w,n} + V_{\hat{w}^h w}^h (\sigma_t^{\hat{w}^h})^\top \sigma_t^{w,n} + \lambda(V^{n,+} - V^h) \Big\}, \end{aligned}$$

where

$$\begin{aligned} \mu_t^{\hat{w}^h} &= r\hat{w}_t^h - \hat{c}_t^h + \hat{w}_t^h \hat{z}_t^{h\top} [\bar{D} + \mu_{P_t}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon], \\ \sigma_t^{\hat{w}^h} &= (\sigma_{P_t}^n + \sigma) \hat{z}_t^h \hat{w}_t^h, \\ \mu_t^{w,n} &= rw_t^n - c_t^n + w_t^n z_t^{n\top} (\bar{D} + \mu_{P_t}^n - rP_t^n), \\ \sigma_t^{w,n} &= (\sigma_{P_t}^n + \sigma) z_t^n w_t^n, \end{aligned}$$

and

$$V^{n,+} = V^n \left(\hat{w}_t^h + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon, w_t^n \right) = \frac{1}{\rho} \log \left(\hat{w}_t^h + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon \right) + \bar{q}^n(w_t^n).$$

Here, $\bar{q}^n(w_t^n)$ is the solution to the ODEs given by Proposition 5.

We conjecture (and verify) that the investor's value function under the hypothetical scenario takes the form

$$V^h(\hat{w}_t^h, w_t^n) = \frac{1}{\rho} \log \hat{w}_t^h + \bar{q}^h(w_t^n).$$

Substituting into the HJB equation yields

$$\begin{aligned} \log \hat{w}_t^h + \rho \bar{q}^h(w_t^n) = \\ \max_{\hat{c}_t^h, \hat{z}_t^h} \left\{ \log \hat{c}_t^h + \frac{1}{\rho \hat{w}_t^h} \mu_t^{\hat{w}^h} - \frac{1}{2\rho(\hat{w}_t^h)^2} (\sigma_t^{\hat{w}^h})^\top \sigma_t^{\hat{w}^h} + \bar{q}^{h'}(w_t^n) \mu_t^{w,n} + \frac{1}{2} \bar{q}^{h''}(w_t^n) (\sigma_t^{w,n})^\top \sigma_t^{w,n} \right. \\ \left. + \lambda \left[\frac{1}{\rho} \log \left(\hat{w}_t^h + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon \right) + \bar{q}^n(w_t^n) - \frac{1}{\rho} \log \hat{w}_t^h - \bar{q}^h(w_t^n) \right] \right\}. \end{aligned}$$

Taking the first-order condition with respect to consumption yields the optimal consumption by the investor under the hypothetical scenario

$$\hat{c}_t^h = \rho \hat{w}_t^h. \quad (\text{A.86})$$

Taking the first-order condition with respect to wealth-scaled asset holdings,

$$0 = \bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon - (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) \hat{z}_t^h + \frac{\text{diag}(\chi)\epsilon}{1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon}.$$

Hence,

$$\hat{z}_t^h = (\Pi_t^n)^{-1} [\bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon] + \frac{1}{1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon} (\Pi_t^n)^{-1} \text{diag}(\chi)\epsilon.$$

From (A.86), the investors' aggregate consumption is given by $c_t^n = \rho w_t^n$. Then we can rewrite the HJB equation as

$$\begin{aligned} \rho \bar{q}^h(w_t^n) = \log \rho + \frac{1}{\rho} \left[r - \rho + \hat{z}_t^{h\top} (\bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon) \right] \\ - \frac{1}{2\rho} \hat{z}_t^{h\top} (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) \hat{z}_t^h + \bar{q}^{h'}(w_t^n) w_t^n \left[r - \rho + \hat{z}_t^{n\top} (\bar{D} + \mu_{Pt}^n - rP_t^n) \right] \\ + \frac{1}{2} \bar{q}^{h''}(w_t^n) (w_t^n)^2 \hat{z}_t^{n\top} (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) \hat{z}_t^n \\ + \lambda \left[\frac{1}{\rho} \log \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon \right) + \bar{q}^n(w_t^n) - \bar{q}^h(w_t^n) \right]. \end{aligned}$$

Note that $\hat{w}_t^h$ has all canceled out.

Since $\hat{y}_t^h = \text{diag}(P_t^n) \hat{z}_t^h$, then

$$\begin{aligned} \hat{y}_t^h &= (\Omega_t^n)^{-1} [\mu_{Rt}^n - r\mathbf{1} + (\mathbf{I} - \text{diag}(\chi))\delta] + \frac{\lambda}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi)\delta} (\Omega_t^n)^{-1} \text{diag}(\chi)\delta \\ &= (\Omega_t^n)^{-1} (\mu_{Rt}^n - r\mathbf{1} + \delta) + \left[\frac{\lambda}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi)\delta} - 1 \right] (\Omega_t^n)^{-1} \text{diag}(\chi)\delta. \end{aligned} \quad (\text{A.87})$$

From (A.18), the investor's demand in the normal state is given by

$$\hat{y}_t^n = (\Omega_t^n)^{-1} (\mu_{Rt}^n - r\mathbf{1}).$$

The dynamic slope is thus

$$\begin{aligned}
\mathcal{C}^{\text{dyn}}(w_t^n; \theta) &= \frac{[\hat{y}_t^h - \hat{y}_t^n]_i}{[\delta]_i} \\
&= \frac{[(\Omega_t^n)^{-1} \delta]_i}{[\delta]_i} + \left[\frac{\lambda}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi) \delta} - 1 \right] \frac{[(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]_i}{[\delta]_i} \\
&= [(\Omega_t^n)^{-1}]_{ii} + \frac{\sum_{j \neq i} [(\Omega_t^n)^{-1}]_{ij} [\delta]_j}{[\delta]_i} + \left[\frac{\lambda}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi) \delta} - 1 \right] \frac{[(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]_i}{[\delta]_i}.
\end{aligned} \tag{A.88}$$

A.5.2 Log Investors: λ Limits of Dynamic Slope

In Section 4.1 when investigating $\mathcal{C}^{\text{log}}$ for the limiting cases $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$, we have derived (24) directly from the corresponding demand curve (23) in Proposition 1. Does (38) converge to (24) in these limits? In the limit of $\lambda \rightarrow \infty$, it is immediate that the last term of (A.88), or the “hedging against reversal” term, goes to zero.

For the $\lambda \rightarrow 0$ case, to ensure consistency we must require that $\chi_i \delta_i = \lambda k_i(\lambda)$ for all i , with an arbitrary bounded function $k(\lambda)$. That is, formally we are taking the limit $\lim_{\lambda \rightarrow 0} \mathcal{C}^{\text{dyn}}(w_t; \lambda, \delta, \text{diag}(\delta)^{-1} \lambda k(\lambda))$ for convergence to (24). The idea is to match the concept of $\mathcal{C}^{\text{dyn}}$ and $\mathcal{M}$ in equilibrium, by requiring $\text{diag}(\chi) \delta = \lambda J_{Rt}^s$, the jump component of $\Delta \mathbb{E}_t(R)$, to hold in equilibrium. Since $\lim_{\lambda \rightarrow 0} J_{Rt}^s$ is bounded in equilibrium, $k(\lambda)$ has to be bounded too when taking the limit of $\mathcal{C}^{\text{log}}$. The last component of (A.88) then equals $\left[\frac{1}{1 + \hat{y}_t^{h\top} k} - 1 \right] \frac{\lambda [(\Omega_t^n)^{-1} k]_i}{[\delta]_i}$, which can be written as

$$-\frac{\hat{y}_t^{h\top} \text{diag}(\chi) \delta}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi) \delta} \frac{[(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]_i}{[\delta]_i}$$

From (A.87),

$$\hat{y}_t^{h\top} = [(\Omega_t^n)^{-1} (\mu_{Rt}^n - r \mathbf{1} + \delta)]^\top - \frac{\hat{y}_t^{h\top} \text{diag}(\chi) \delta}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi) \delta} [(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]^\top.$$

Multiplying both sides by $\text{diag}(\chi) \delta$ yields

$$\begin{aligned}
\hat{y}_t^{h\top} \text{diag}(\chi) \delta &= \underbrace{[(\Omega_t^n)^{-1} (\mu_{Rt}^n - r \mathbf{1} + \delta)]^\top \text{diag}(\chi) \delta}_{p(\lambda)} \\
&\quad - \frac{\hat{y}_t^{h\top} \text{diag}(\chi) \delta}{\lambda + \hat{y}_t^{h\top} \text{diag}(\chi) \delta} \underbrace{[(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]^\top \text{diag}(\chi) \delta}_{q(\lambda)}.
\end{aligned}$$

Then

$$\hat{y}_t^{h\top} \text{diag}(\chi) \delta = \frac{-(\lambda - q(\lambda) - p(\lambda)) + \sqrt{(\lambda - q(\lambda) - p(\lambda))^2 + 4p(\lambda)\lambda}}{2}.$$

Then as $\lambda \rightarrow 0$,

$$\hat{y}_t^{h\top} \text{diag}(\chi) \delta \rightarrow p(\lambda) + q(\lambda) = \lambda \left[[(\Omega_t^n)^{-1} (\mu_{Rt}^n - r\mathbf{1} + \delta)]^\top k + \lambda [(\Omega_t^n)^{-1} k]^\top k \right] \rightarrow 0$$

A.5.3 CRRA Investors

The individual investor's HJB equation is

$$\begin{aligned} \rho V^h = \max_{\hat{c}_t^h, \hat{z}_t^h} & \left\{ \frac{(\hat{c}_t^h)^{1-\gamma}}{1-\gamma} + V_{\hat{w}^h}^h \mu_t^{\hat{w}^h} + \frac{1}{2} V_{\hat{w}^h \hat{w}^h}^h (\sigma_t^{\hat{w}^h})^\top \sigma_t^{\hat{w}^h} + V_w^h \mu_t^{w,n} \right. \\ & \left. + \frac{1}{2} V_{ww}^h (\sigma_t^{w,n})^\top \sigma_t^{w,n} + V_{\hat{w}^h w}^h (\sigma_t^{\hat{w}^h})^\top \sigma_t^{w,n} + \lambda (V^{n,+} - V^h) \right\}, \end{aligned}$$

where

$$\begin{aligned} \mu_t^{\hat{w}^h} &= r\hat{w}_t^h - \hat{c}_t^h + \hat{w}_t^h \hat{z}_t^{h\top} [\bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon], \\ \sigma_t^{\hat{w}^h} &= (\sigma_{Pt}^n + \sigma) \hat{z}_t^h \hat{w}_t^h, \\ \mu_t^{w,n} &= rw_t^n - c_t^n + w_t^n z_t^{n\top} (\bar{D} + \mu_{Pt}^n - rP_t^n), \\ \sigma_t^{w,n} &= (\sigma_{Pt}^n + \sigma) z_t^n w_t^n, \end{aligned}$$

and

$$V^{n,+} = V^n \left(\hat{w}_t^h + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon, w_t^n \right) = q^n(w_t^n) \frac{(\hat{w}_t^h + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon)^{1-\gamma}}{1-\gamma}.$$

Here, $q^n(w_t^n)$ is the solution to the ODEs given by Proposition 5.

We conjecture (and verify) that the investor's value function takes the form

$$V^h(\hat{w}_t^h, w_t^n) = q^h(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma}.$$

Substituting into the HJB equation

$$\begin{aligned} & \rho q^h(w_t^n; \lambda, \delta, \chi) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} \\ &= \max_{\hat{c}_t^h, \hat{z}_t^h} \left\{ \frac{(\hat{c}_t^h)^{1-\gamma}}{1-\gamma} + q^h(w_t^n; \theta) (\hat{w}_t^h)^{-\gamma} \mu_t^{\hat{w}^h} - \frac{\gamma}{2} q^h(w_t^n; \theta) (\hat{w}_t^h)^{-\gamma-1} (\sigma_t^{\hat{w}^h})^\top \sigma_t^{\hat{w}^h} \right. \\ & \quad + q^{h'}(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} \mu_t^{w,n} + \frac{1}{2} q^{h''}(w_t^n; \lambda, \delta, \chi) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} (\sigma_t^{w,n})^\top \sigma_t^{w,n} \\ & \quad + q^{h'}(w_t^n; \lambda, \delta, \chi) (\hat{w}_t^h)^{-\gamma} (\sigma_t^{\hat{w}^h})^\top \sigma_t^{w,n} \\ & \quad \left. + \lambda \left[q^n(w_t^n) \frac{(\hat{w}_t^h + \frac{1}{\lambda} \hat{w}_t^h \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon)^{1-\gamma}}{1-\gamma} - q^h(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} \right] \right\}. \end{aligned}$$

Taking the first-order condition with respect to consumption yields the optimal consumption

by the investor as

$$\hat{c}_t^h = q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} \hat{w}_t^h$$

Taking the first-order condition with respect to wealth-scaled asset holdings,

$$\begin{aligned} 0 = & q^h(w_t^n; \lambda, \delta, \chi) [\bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon] \\ & - \gamma q^h(w_t^n; \lambda, \delta, \chi) (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) \hat{z}_t^h \\ & + q^{h'}(w_t^n; \theta) (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) Y_t^n + q^n(w_t^n) \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon\right)^{-\gamma} \text{diag}(\chi)\epsilon. \end{aligned}$$

Then

$$\begin{aligned} \hat{z}_t^h = & \frac{1}{\gamma} (\Pi_t^n)^{-1} [\bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon] + \frac{1}{\gamma} \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \lambda, \delta, \chi)} Y_t^n \\ & + \frac{1}{\gamma} \frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} (\Pi_t^n)^{-1} \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon\right)^{-\gamma} \text{diag}(\chi)\epsilon. \end{aligned}$$

Moreover, investors' aggregate consumption is given by

$$c_t^n = q^n(w_t^n)^{-\frac{1}{\gamma}} w_t^n.$$

Then we can rewrite the HJB equation as

$$\begin{aligned} \rho q^h(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} = & q^h(w_t^n; \lambda, \delta, \chi)^{1-\frac{1}{\gamma}} \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} \\ & + q^h(w_t^n; \theta) (\hat{w}_t^h)^{1-\gamma} \left[r - q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} + \hat{z}_t^{h\top} (\bar{D} + \mu_{Pt}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon) \right] \\ & - \frac{\gamma}{2} q^h(w_t^n; \theta) (\hat{w}_t^h)^{1-\gamma} \hat{z}_t^{h\top} (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) \hat{z}_t^h \\ & + w_t^n q^{h'}(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} \left[r - q^n(w_t^n)^{-\frac{1}{\gamma}} + z_t^{n\top} (\bar{D} + \mu_{Pt}^n - rP_t^n) \right] \\ & + \frac{1}{2} (w_t^n)^2 q^{h''}(w_t^n; \lambda, \delta, \chi) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} z_t^{n\top} (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) z_t^n \\ & + w_t^n q^{h'}(w_t^n; \lambda, \delta, \chi) (\hat{w}_t^h)^{1-\gamma} \hat{z}_t^{h\top} (\sigma_{Pt}^n + \sigma)^\top (\sigma_{Pt}^n + \sigma) z_t^n \\ & + \lambda \left[q^n(w_t^n) \frac{(\hat{w}_t^h)^{1-\gamma} (1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon)^{1-\gamma}}{1-\gamma} - q^h(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma} \right]. \end{aligned}$$

Dividing both sides by $q^h(w_t^n; \theta) \frac{(\hat{w}_t^h)^{1-\gamma}}{1-\gamma}$ yields

$$\begin{aligned}
& q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} + (1-\gamma) \left[r - q^h(w_t^n; \lambda, \delta, \chi)^{-\frac{1}{\gamma}} + \hat{z}_t^{h\top} (\bar{D} + \mu_{P_t}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon) \right] \\
& + w_t^n \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \left[r - q^n(w_t^n)^{-\frac{1}{\gamma}} + z_t^{n\top} (\bar{D} + \mu_{P_t}^n - rP_t^n) \right] \\
& - \frac{\gamma(1-\gamma)}{2} \hat{z}_t^{h\top} (\sigma_{P_t}^n + \sigma)^\top (\sigma_{P_t}^n + \sigma) \hat{z}_t^h \\
& + \frac{1}{2} (w_t^n)^2 \frac{q^{h''}(w_t^n; \lambda, \delta, \chi)}{q^h(w_t^n; \theta)} z_t^{n\top} (\sigma_{P_t}^n + \sigma)^\top (\sigma_{P_t}^n + \sigma) z_t^n \\
& + (1-\gamma) w_t^n \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \hat{z}_t^{h\top} (\sigma_{P_t}^n + \sigma)^\top (\sigma_{P_t}^n + \sigma) z_t^n \\
& + \lambda \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \lambda, \delta, \chi)} \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon \right)^{1-\gamma} - 1 \right] = \rho
\end{aligned}$$

Rearranging yields

$$\begin{aligned}
& q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} \\
& + (1-\gamma) \left[r - q^h(w_t^n; \lambda, \delta, \chi)^{-\frac{1}{\gamma}} + \hat{z}_t^{h\top} (\bar{D} + \mu_{P_t}^n - rP_t^n + (\mathbf{I} - \text{diag}(\chi))\epsilon) \right] \\
& + w_t^n \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \left[r - q^n(w_t^n)^{-\frac{1}{\gamma}} + z_t^{n\top} (\bar{D} + \mu_{P_t}^n - rP_t^n) + (1-\gamma) \hat{z}_t^{h\top} \Pi_t^n z_t^n \right] \\
& - \frac{\gamma(1-\gamma)}{2} \hat{z}_t^{h\top} \Pi_t^n \hat{z}_t^h + \frac{1}{2} (w_t^n)^2 \frac{q^{h''}(w_t^n; \theta)}{q^h(w_t^n; \theta)} z_t^{n\top} \Pi_t^n z_t^n \\
& + \lambda \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi)\epsilon \right)^{1-\gamma} - 1 \right] = \rho.
\end{aligned}$$

Since

$$\begin{aligned}
\bar{D} + \mu_{P_t}^n - rP_t^n &= \frac{\Gamma^n(w_t^n) \hat{\sigma}_t^{n\top} \sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2}, \\
z_t^{n\top} (\bar{D} + \mu_{P_t}^n - rP_t^n) &= \frac{\Gamma^n(w_t^n) z_t^{n\top} \Sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2}, \\
\sigma_{P_t}^n + \sigma &= \frac{\hat{\sigma}_t^n}{1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n},
\end{aligned}$$

where

$$\hat{\sigma}_t^n \equiv -w_t^n \sigma z_t^n \pi^{n'}(w_t^n)^\top + [1 + w_t^n \pi^{n'}(w_t^n)^\top z_t^n] \sigma.$$

Hence, the ODE becomes

$$\begin{aligned}
& q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} + (1 - \gamma) \left[r - q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} + \hat{z}^{h\top} \left(\frac{\Gamma^n(w_t^n) \hat{\sigma}_t^{n\top} \sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} + (\mathbf{I} - \text{diag}(\chi)) \epsilon \right) \right] \\
& + w_t^n \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \left[r - q^n(w_t^n)^{-\frac{1}{\gamma}} + \frac{\Gamma^n(w_t^n) z_t^{n\top} \Sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} + (1 - \gamma) \frac{\hat{z}_t^{h\top} \hat{\sigma}_t^{n\top} \sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} \right] \\
& - \frac{\gamma(1 - \gamma)}{2} \frac{\hat{z}_t^{h\top} \hat{\sigma}_t^{n\top} \hat{\sigma}_t^n \hat{z}_t^h}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} + \frac{1}{2} (w_t^n)^2 \frac{q^{h''}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \frac{z_t^{n\top} \Sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} \\
& + \lambda \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \frac{1}{\lambda} \hat{z}^{h\top} \text{diag}(\chi) \epsilon \right)^{1-\gamma} - 1 \right] = \rho.
\end{aligned} \tag{A.89}$$

Moreover,

$$\begin{aligned}
\hat{z}_t^h &= \frac{1}{\gamma} (\hat{\sigma}_t^{n\top} \hat{\sigma}_t^n)^{-1} \left\{ \Gamma^n(w_t^n) \hat{\sigma}_t^{n\top} \sigma z_t^n + [1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2 (\mathbf{I} - \text{diag}(\chi)) \epsilon \right\} \\
& + \frac{q^{h'}(w_t^n; \theta)}{\gamma q^h(w_t^n; \theta)} z_t^n w_t^n \\
& + \frac{1}{\gamma} \frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} [1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2 (\hat{\sigma}_t^{n\top} \hat{\sigma}_t^n)^{-1} \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon \right)^{-\gamma} \text{diag}(\chi) \epsilon.
\end{aligned}$$

That is,

$$\begin{aligned}
\hat{z}_t^h &= \left[\frac{1}{\gamma} \Gamma^n(w_t^n) + \frac{q^{h'}(w_t^n; \theta)}{\gamma q^h(w_t^n; \theta)} w_t^n \right] z_t^n + \frac{1}{\gamma} [1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2 (\hat{\sigma}_t^{n\top} \hat{\sigma}_t^n)^{-1} (\mathbf{I} - \text{diag}(\chi)) \epsilon \\
& + \frac{1}{\gamma} \frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} [1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2 (\hat{\sigma}_t^{n\top} \hat{\sigma}_t^n)^{-1} \left(1 + \frac{1}{\lambda} \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon \right)^{-\gamma} \text{diag}(\chi) \epsilon.
\end{aligned}$$

□

A.6 Proof of Proposition 7

With (34), we can restate (40) as

$$\begin{aligned}
\mathcal{C}^{\text{dyn}}(w_t^n; \theta) &= \frac{[\hat{y}_t^h - \hat{y}_t^n]_i}{\delta_i} = \mathcal{C}^{\text{myo}}(w_t^n; \delta) \\
& + \frac{1}{\gamma} \left[\frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} - \frac{q^{n'}(w_t^n)}{q^n(w_t^n)} \right] \frac{[y_t^n w_t^n]_i}{[\delta]_i} \\
& + \frac{1}{\gamma} \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \hat{y}_t^{h\top} \frac{\text{diag}(\chi) \delta}{\lambda} \right)^{-\gamma} - 1 \right] \frac{[(\Omega_t^n)^{-1} \text{diag}(\chi) \delta]_i}{[\delta]_i}.
\end{aligned} \tag{A.90}$$

First, we show that the second component (Brownian hedging) of (A.90) goes to zero as

$\lambda \rightarrow \infty$. Equivalently, we want to show that

$$\lim_{\lambda \rightarrow \infty} \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} - \frac{q^{n'}(w_t^n)}{q^n(w_t^n)} = 0.$$

We only need to show that

$$\left(\frac{q^h(w_t^n; \theta)}{q^n(w_t^n)} \right)'_{w^n} = 0,$$

that is the limiting ratio does not depend on w_t^n .

Restating the ODE (A.89) as

$$\begin{aligned} & q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} + (1 - \gamma) \left[r - q^h(w_t^n; \theta)^{-\frac{1}{\gamma}} + \hat{z}^{h\top} \left(\frac{\Gamma^n(w_t^n) \hat{\sigma}_t^{n\top} \sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} + (\mathbf{I} - \text{diag}(\chi)) \epsilon \right) \right] \\ & + w_t^n \frac{q^{h'}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \left[r - q^n(w_t^n)^{-\frac{1}{\gamma}} + \frac{\Gamma^n(w_t^n) z_t^{n\top} \Sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} + (1 - \gamma) \frac{\hat{z}_t^{h\top} \hat{\sigma}_t^{n\top} \sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} \right] \\ & - \frac{\gamma(1 - \gamma)}{2} \frac{\hat{z}_t^{h\top} \hat{\sigma}_t^{n\top} \hat{\sigma}_t^n \hat{z}_t^h}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} + \frac{1}{2} (w_t^n)^2 \frac{q^{h''}(w_t^n; \theta)}{q^h(w_t^n; \theta)} \frac{z_t^{n\top} \Sigma z_t^n}{[1 + \pi^{n'}(w_t^n)^\top z_t^n w_t^n]^2} \\ & + \lambda \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \frac{1}{\lambda} \hat{z}^{h\top} \text{diag}(\chi) \epsilon \right)^{1-\gamma} - 1 \right] = \rho. \end{aligned}$$

Define $\varepsilon \equiv 1/\lambda$. Then the left-hand side of the ODE (A.89) can be rewritten as the sum of a collection of terms that are independent of ε and the jump term that depends on ε . That is,

$$\frac{1}{\varepsilon} \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \varepsilon \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon \right)^{1-\gamma} - 1 \right] + O(1) = \rho. \quad (\text{A.91})$$

For small ε (or equivalently, large λ), we can apply the following binomial expansion

$$\left(1 + \varepsilon \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon \right)^{1-\gamma} = 1 + (1 - \gamma) \varepsilon \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon + O(\varepsilon^2).$$

Thus, the first term on the left-hand side of (A.91) can be written as

$$\begin{aligned} & \frac{1}{\varepsilon} \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + (1 - \gamma) \varepsilon \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon + O(\varepsilon^2) \right) - 1 \right] \\ & = \frac{1}{\varepsilon} \left[\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} - 1 \right] + (1 - \gamma) \frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \hat{z}_t^{h\top} \text{diag}(\chi) \epsilon + O(\varepsilon). \end{aligned}$$

When $\lambda \rightarrow \infty$, $\varepsilon \rightarrow 0$. The only way the left-hand side of the ODE does not diverge as $\varepsilon \rightarrow 0$ is that

$$\frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} - 1 = O(\varepsilon).$$

Therefore,

$$\lim_{\lambda \rightarrow \infty} \frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} = 1 \quad (\text{A.92})$$

for all w_t^n . Thus, the limiting ratio is a constant and does not depend on w_t^n . Hence, we have proved that the Brownian hedging component of (A.90) goes to zero as $\lambda \rightarrow \infty$.

Next, we want to show that the last component (hedging against reversal) also goes to zero when $\lambda \rightarrow \infty$. Equivalently, we want to show that

$$\lim_{\lambda \rightarrow \infty} \frac{q^n(w_t^n)}{q^h(w_t^n; \theta)} \left(1 + \hat{y}_t^{h\top} \frac{\text{diag}(\chi)\delta}{\lambda} \right)^{-\gamma} - 1 = 0. \quad (\text{A.93})$$

Given χ and δ ,

$$\lim_{\lambda \rightarrow \infty} \left(1 + \hat{y}_t^{h\top} \frac{\text{diag}(\chi)\delta}{\lambda} \right)^{-\gamma} = 1.$$

Combined with (A.92), it is thus immediate that (A.93) holds. Therefore,

$$\lim_{\lambda \rightarrow \infty} \mathcal{C}^{\text{dyn}}(w_t^n; \theta) = \mathcal{C}^{\text{myo}}(w_t^n; \delta).$$

□

B Connection to HHHKL

In this appendix, we show that the demand curve (45) derived in Section 6.2 locally has the form HHHKL assumes. That is, the HHHKL estimation procedure, if one finds instruments satisfying the standard conditions, still remains valid.

The demand shifter in (45) is given by

$$e_t^\xi = \frac{w_t^\xi q^{\xi'}(w_t^\xi)}{\gamma q^\xi(w_t^\xi)} Y_t^\xi + \frac{w_t^\xi}{\gamma} (\Pi_t^\xi)^{-1} \left[\bar{D} + \mu_{Pt}^\xi + \frac{\lambda^\xi q^{-\xi}(w_t^\xi + Y_t^{\xi\top} J_t^\xi)}{q^\xi(w_t^\xi) \left(1 + \frac{1}{\hat{w}_t^\xi} \hat{Y}_t^{\xi\top} J_t^\xi \right)^\gamma} J_t^\xi \right], \quad (\text{B.94})$$

where $\Pi_t^\xi \equiv (\sigma_{Pt}^\xi + \sigma)^\top (\sigma_{Pt}^\xi + \sigma)$ is the variance-covariance matrix of dollar returns and $J_t^\xi \equiv P_t^{-\xi} - P_t^\xi$ is the jump in the price vector as the economy transitions from state ξ to state $-\xi$.

For simplicity, we first consider i.i.d. underlying cash flow shocks, so that the exogenous diffusion matrix in (6) for the I assets is $\sigma = \bar{\sigma} \mathbf{I}$, with $\bar{\sigma}$ being a scalar and $\mathbf{I}$ an $I \times I$ identity matrix. (We consider a factor structure for σ in Section B.1.) Following the same logic as in (27) in Section 4.2, we can derive the endogenous dollar return diffusion matrix of the assets

to be

$$\underbrace{\sigma_{P_t}^\xi}_{I \times I} = \frac{\bar{\sigma}}{1 - \underbrace{P^{\xi'}(w_t^\xi)^\top Y_t^\xi}_{\text{scalar}}} \cdot \underbrace{Y_t^\xi}_{I \times 1} \underbrace{P^{\xi'}(w_t^\xi)^\top}_{1 \times I}.$$

Then

$$\sigma_{P_t}^\xi + \sigma = \bar{\sigma} \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right].$$

Hence, the endogenous variance-covariance matrix of dollar returns can be written as

$$\begin{aligned} \Pi_t^\xi &= (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \\ &= \bar{\sigma}^2 \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right]^\top \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right] \\ &= \bar{\sigma}^2 \left[\mathbf{I} + \frac{P^{\xi'}(w_t^\xi) Y_t^{\xi\top}}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right] \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right]. \end{aligned}$$

Taking the inverse of Π_t^ξ and using the Sherman-Morrison formula, we get

$$\begin{aligned} (\Pi_t^\xi)^{-1} &= \frac{1}{\bar{\sigma}^2} \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right]^{-1} \left[\mathbf{I} + \frac{P^{\xi'}(w_t^\xi) Y_t^{\xi\top}}{1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi} \right]^{-1} \\ &= \frac{1}{\bar{\sigma}^2} \left[\mathbf{I} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top \right] \left[\mathbf{I} - P^{\xi'}(w_t^\xi) Y_t^{\xi\top} \right] \\ &= \frac{1}{\bar{\sigma}^2} \left[\mathbf{I} - P^{\xi'}(w_t^\xi) Y_t^{\xi\top} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top + P^{\xi'}(w_t^\xi)^\top P^{\xi'}(w_t^\xi) Y_t^\xi Y_t^{\xi\top} \right]. \end{aligned}$$

We can define the asset characteristic matrix

$$\beta \equiv \begin{bmatrix} Y_t^\xi & P^{\xi'}(w_t^\xi) \end{bmatrix}_{I \times 2} \quad (\text{B.95})$$

and the substitution matrix

$$\mathcal{E}_{sub} \equiv \begin{bmatrix} P^{\xi'}(w_t^\xi)^\top P^{\xi'}(w_t^\xi) & -1 \\ -1 & 0 \end{bmatrix}_{2 \times 2}$$

Then it is easy to verify that

$$(\Pi_t^\xi)^{-1} = \frac{1}{\bar{\sigma}^2} \left(\mathbf{I} + \beta \mathcal{E}_{sub} \beta^\top \right)$$

Substituting into the investor's individual demand (45), we get

$$\begin{aligned}\hat{Y}_t^\xi &= \frac{w_t^\xi}{\gamma \bar{\sigma}^2} (\mathbf{I} + \beta \mathcal{E}_{sub} \beta^\top) (\bar{D} + \mu_{Pt}^\xi - r P_t^\xi) + \frac{w_t^\xi}{\gamma} \frac{q^{\xi'}(w_t^\xi)}{q^\xi(w_t^\xi)} Y_t^\xi \\ &\quad + \frac{\lambda^\xi w_t^\xi}{\gamma} (\Pi_t^\xi)^{-1} \frac{q^{-\xi}(w_t^\xi + Y_t^{\xi\top} J_t^\xi)}{q^\xi(w_t^\xi) \left(1 + \frac{1}{\hat{w}_t^\xi} \hat{Y}_t^{\xi\top} J_t^\xi\right)^\gamma} J_t^\xi.\end{aligned}$$

Hence, holding the demand shifter e_t in (45) fixed, the price-coefficient matrix of the demand system takes the exact form studied in HHHKL:

$$\mathcal{E}_t \equiv \left[\frac{\partial \hat{Y}^\xi}{\partial P^\xi} \right]_{I \times I} = - \frac{r w_t^\xi}{\gamma \bar{\sigma}^2} \left(\mathbf{I} + \beta \mathcal{E}_{sub} \beta^\top \right).$$

Note that $\mathcal{E}_t$ is a partial rather than a total derivative with respect to P^ξ : the shifter e_t itself depends on P^ξ through J_t^ξ .

In the language of HHHKL, in our model each asset has two (endogenous) characteristics, as suggested by (B.95): its aggregate holdings by investors Y_t^ξ , and its price loading on the aggregate investor wealth $P^{\xi'}(w_t^\xi)$. So far we have assumed away any exogenous factor structure for the dividend process; if we were to assume a K -factor structure for the dividend diffusion σ , the total dimension of asset characteristics would be $K + 2$, as we show next.

B.1 Factor Structure for Variance-Covariance Matrix

We now add another factor structure for the variance-covariance matrix of cash flows. Specifically, we suppose

$$\Sigma \equiv \sigma^\top \sigma = \bar{\sigma}^2 (\mathbf{I} + \psi \psi^\top),$$

where $\bar{\sigma}$ is a scalar, $\mathbf{I}$ is an $I \times I$ identity matrix and ψ is $I \times K$. By eigen-decomposition, the diffusion σ can be written as

$$\sigma = \bar{\sigma} (\mathbf{I} + \nu \Lambda \nu^\top),$$

where ν are the eigenvectors of $\psi \psi^\top$ corresponding to its non-zero eigenvalues (size $I \times K$), and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_K)$ where $\lambda_i = -1 + \sqrt{1 + s_i^2}$ where s_i^2 denotes eigenvalues of $\psi \psi^\top$.

Let $A \equiv \mathbf{I} + \nu \Lambda \nu^\top$, so that $\sigma = \bar{\sigma} A$ and $\sigma^\top \sigma = \bar{\sigma}^2 A^\top A = \Sigma$.

The price diffusion solves the fixed point

$$\sigma_{Pt}^\xi = (\sigma_{Pt}^\xi + \sigma) Y_t^\xi P^{\xi'}(w_t^\xi)^\top,$$

which can be rearranged to obtain

$$\sigma_{Pt}^\xi + \sigma = \sigma \left[\mathbf{I} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top \right]^{-1} = \bar{\sigma} A \left[\mathbf{I} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top \right]^{-1}.$$

Writing $d \equiv 1 - P^{\xi'}(w_t^\xi)^\top Y_t^\xi$ and applying the Sherman-Morrison formula,

$$\sigma_{P_t}^\xi + \sigma = \bar{\sigma} A \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{d} \right].$$

Equilibrium requires $d > 0$. The endogenous variance-covariance matrix of dollar returns can thus be written as

$$\begin{aligned} \Pi_t^\xi &\equiv (\sigma_{P_t}^\xi + \sigma)^\top (\sigma_{P_t}^\xi + \sigma) \\ &= \bar{\sigma}^2 \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{d} \right]^\top (A^\top A) \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{d} \right]. \end{aligned}$$

Taking the inverse of Π_t^ξ yields

$$(\Pi_t^\xi)^{-1} = \frac{1}{\bar{\sigma}^2} \left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{d} \right]^{-1} (A^\top A)^{-1} \left[\mathbf{I} + \frac{P^{\xi'}(w_t^\xi) Y_t^{\xi\top}}{d} \right]^{-1}.$$

By the Sherman-Morrison formula, and using $d + P^{\xi'}(w_t^\xi)^\top Y_t^\xi = 1$, we have

$$\left[\mathbf{I} + \frac{Y_t^\xi P^{\xi'}(w_t^\xi)^\top}{d} \right]^{-1} = \mathbf{I} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top,$$

and similarly,

$$\left[\mathbf{I} + \frac{P^{\xi'}(w_t^\xi) Y_t^{\xi\top}}{d} \right]^{-1} = \mathbf{I} - P^{\xi'}(w_t^\xi) Y_t^{\xi\top}.$$

Hence,

$$(\Pi_t^\xi)^{-1} = \frac{1}{\bar{\sigma}^2} \left[\mathbf{I} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top \right] (A^\top A)^{-1} \left[\mathbf{I} - P^{\xi'}(w_t^\xi) Y_t^{\xi\top} \right]. \quad (\text{B.96})$$

It remains to invert $A^\top A$ which can be written as

$$A^\top A = \mathbf{I} + \nu(2\Lambda + \Lambda\Lambda)\nu^\top = \mathbf{I} + \nu \text{diag}(s_1^2, \dots, s_K^2)\nu^\top.$$

Applying the Woodbury identity and using the fact that $\nu^\top \nu = \mathbf{I}$, one can show that

$$\begin{aligned} (A^\top A)^{-1} &= \mathbf{I} - \nu \left(\text{diag}(1/s_1^2, \dots, 1/s_K^2) + \mathbf{I} \right)^{-1} \nu^\top \\ &= \mathbf{I} - \nu \text{diag} \left(\frac{s_1^2}{1 + s_1^2}, \dots, \frac{s_K^2}{1 + s_K^2} \right) \nu^\top. \end{aligned}$$

Hence, we can write $(A^\top A)^{-1}$ as

$$(A^\top A)^{-1} = \mathbf{I} + \nu M \nu^\top, \quad (\text{B.97})$$

where

$$M \equiv \text{diag} \left(-\frac{s_1^2}{1+s_1^2}, \dots, -\frac{s_K^2}{1+s_K^2} \right) = [(\mathbf{I} + \Lambda)(\mathbf{I} + \Lambda)]^{-1} - \mathbf{I}.$$

Substituting (B.97) into (B.96), we get

$$\begin{aligned} \bar{\sigma}^2(\Pi_t^\xi)^{-1} &= \left[\mathbf{I} - Y_t^\xi P^{\xi'}(w_t^\xi)^\top \right] \left[\mathbf{I} + \nu M \nu^\top \right] \left[\mathbf{I} - P^{\xi'}(w_t^\xi) Y_t^{\xi\top} \right] \\ &= \mathbf{I} + \nu M \nu^\top - Y_t^\xi P^{\xi'}(w_t^\xi)^\top - Y_t^\xi P^{\xi'}(w_t^\xi)^\top \nu M \nu^\top - P^{\xi'}(w_t^\xi) Y_t^{\xi\top} \\ &\quad - \nu M \nu^\top P^{\xi'}(w_t^\xi) Y_t^{\xi\top} + Y_t^\xi P_t^{\xi'}(w_t^\xi)^\top P^{\xi'}(w_t^\xi) Y_t^{\xi\top} + Y_t^\xi P_t^{\xi'}(w_t^\xi)^\top \nu M \nu^\top P^{\xi'}(w_t^\xi) Y_t^{\xi\top}. \end{aligned}$$

Let us define

$$\beta^{full} \equiv \begin{bmatrix} \nu & Y_t^\xi & P^{\xi'}(w_t^\xi) \end{bmatrix}_{I \times (K+2)}$$

and

$$\mathcal{E}^{full} \equiv \begin{bmatrix} M & -M \nu^\top P^{\xi'}(w_t^\xi) & 0 \\ -P^{\xi'}(w_t^\xi)^\top \nu M & P^{\xi'}(w_t^\xi)^\top P^{\xi'}(w_t^\xi) + P^{\xi'}(w_t^\xi)^\top \nu M \nu^\top P^{\xi'}(w_t^\xi) & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

which is of size $(K+2) \times (K+2)$. Then

$$(\Pi_t^\xi)^{-1} = \frac{1}{\bar{\sigma}^2} \left(\mathbf{I} + \beta^{full} \mathcal{E}^{full} \beta^{full\top} \right).$$

C The CARA case

We also consider the case with CARA investors. For these investors, their value functions are independent of the investors' total wealth w_t . Thus, w_t is not a state variable in this case. An investor maximizes

$$-\mathbb{E}_t \left[\int_t^\infty e^{-\rho(\tau-t)} e^{-\frac{\gamma}{r} \hat{c}_\tau} d\tau \right], \quad (\text{C.98})$$

subject to the wealth dynamic

$$d\hat{w}_t^\xi = r\hat{w}_t^\xi dt + \hat{Y}_t^{\xi\top} (dD_t - rP_t^\xi dt) + \hat{Y}_t^{\xi\top} J_t^\xi dN_t^\xi - \hat{c}_t^\xi dt. \quad (\text{C.99})$$

Note that the asset prices depend only on the state of the economy ξ . That is, in a given state of the economy, the asset prices are constant. Thus, the investor's dollar excess returns from investing in the risky assets become $dD_t - rP_t^\xi dt$. Substituting (6) into the investor's wealth dynamic (C.99) yields

$$d\hat{w}_t^\xi = \left[r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{Y}_t^{\xi\top} (\bar{D} - rP_t^\xi) \right] dt + \hat{Y}_t^{\xi\top} \sigma^\top dB_t + \hat{Y}_t^{\xi\top} J_t^\xi dN_t^\xi. \quad (\text{C.100})$$

The investor's problem is to choose the optimal consumption and asset holdings $\{\hat{c}_t^\xi, \hat{Y}_t^\xi\}$ to maximize their utility (C.98) subject to the wealth dynamic (C.100) while taking prices P_t^ξ as given. Let $V^\xi(\hat{w}_t^\xi)$ denote the investor's value function in state ξ . The investor's HJB equation can be written as

$$\rho V^\xi = \max_{\hat{c}_t^\xi, \hat{Y}_t^\xi} \left\{ -e^{-\frac{\gamma}{r}\hat{c}_t^\xi} + V_{\hat{w}}^\xi \mu_t^{\hat{w}, \xi} + \frac{1}{2} V_{\hat{w}\hat{w}}^\xi (\sigma_t^{\hat{w}, \xi})^\top \sigma_t^{\hat{w}, \xi} + \lambda^\xi (V^{-\xi} - V^\xi) \right\},$$

where

$$\begin{aligned} \mu_t^{\hat{w}, \xi} &= r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{Y}_t^{\xi\top} (\bar{D} - rP_t^\xi), \\ \sigma_t^{\hat{w}, \xi} &= \sigma \hat{Y}_t^\xi. \end{aligned}$$

We conjecture (and later verify) that the investor's value function takes the form

$$V^\xi(\hat{w}_t^\xi) = -e^{-\gamma\hat{w}_t^\xi} q^\xi,$$

where q^ξ is a constant. The long-term CARA investor's value function over wealth has the same negative-exponential form as the utility function over consumption, with the risk-aversion coefficient being γ rather than γ/r . Substituting the conjectured value function form into the HJB equation, we can rewrite the HJB equation as

$$\begin{aligned} -\rho e^{-\gamma\hat{w}_t^\xi} q^\xi &= \max_{\hat{c}_t^\xi, \hat{Y}_t^\xi} \left\{ -e^{-\frac{\gamma}{r}\hat{c}_t^\xi} + \gamma e^{-\gamma\hat{w}_t^\xi} q^\xi \left[r\hat{w}_t^\xi - \hat{c}_t^\xi + \hat{Y}_t^{\xi\top} (\bar{D} - rP_t^\xi) \right] \right. \\ &\quad \left. - \frac{1}{2} \gamma^2 e^{-\gamma\hat{w}_t^\xi} q^\xi \hat{Y}_t^{\xi\top} \Sigma \hat{Y}_t^\xi + \lambda^\xi \left[-e^{-\gamma[\hat{w}_t^\xi + \hat{Y}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]} q^{-\xi} + e^{-\gamma\hat{w}_t^\xi} q^\xi \right] \right\}. \end{aligned}$$

Taking the first-order condition with respect to $\hat{c}_t$ yields

$$\frac{\gamma}{r} e^{-\frac{\gamma}{r}\hat{c}_t^\xi} = \gamma e^{-\gamma\hat{w}_t^\xi} q^\xi,$$

which implies that the investor's optimal consumption $\hat{c}_t^\xi$ at time t if the economy is in state ξ is given by

$$\hat{c}_t^\xi = r\hat{w}_t^\xi - \frac{r}{\gamma} (\log q^\xi + \log r). \quad (\text{C.101})$$

Taking the first-order condition with respect to $\hat{Y}_t^\xi$ yields

$$\begin{aligned} \gamma e^{-\gamma\hat{w}_t^\xi} q^\xi (\bar{D} - rP_t^\xi) - \gamma^2 e^{-\gamma\hat{w}_t^\xi} q^\xi \Sigma \hat{Y}_t^\xi + \lambda^\xi \gamma e^{-\gamma[\hat{w}_t^\xi + \hat{Y}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)]} q^{-\xi} (P_t^{-\xi} - P_t^\xi) \\ = 0. \end{aligned}$$

Rearranging, we obtain the investor's optimal asset holdings $\hat{Y}_t^\xi$ as

$$\hat{Y}_t^\xi = \frac{1}{\gamma} \Sigma^{-1} (\bar{D} - rP_t^\xi) + \frac{\lambda^\xi q^{-\xi}}{\gamma q^\xi} e^{-\gamma\hat{Y}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \Sigma^{-1} (P_t^{-\xi} - P_t^\xi). \quad (\text{C.102})$$

The investor's individual demand can be written as

$$\begin{aligned}\hat{Y}_t^\xi = & \frac{1}{\gamma} \Sigma^{-1} [\bar{D} - rP_t^\xi + \lambda^\xi (P_t^{-\xi} - P_t^\xi)] \\ & + \frac{\lambda^\xi}{\gamma} \left[\frac{q^{-\xi}}{q^\xi} e^{-\gamma \hat{Y}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} - 1 \right] \Sigma^{-1} (P_t^{-\xi} - P_t^\xi),\end{aligned}$$

where $\bar{D} - rP_t^\xi + \lambda^\xi (P_t^{-\xi} - P_t^\xi)$ corresponds to the investor's expected dollar excess returns at time t when the economy is in state ξ .

Substituting the investor's optimal consumption $\hat{c}_t^\xi$ given by (C.101) and the investor's optimal asset holdings $\hat{Y}_t^\xi$ given by (C.102) into the HJB equation, we have

$$\begin{aligned}-\rho = & r(\log q^\xi + \log r - 1) + \gamma \hat{Y}_t^{\xi\top} (\bar{D} - rP_t^\xi) - \frac{1}{2} \gamma^2 \hat{Y}_t^{\xi\top} \Sigma \hat{Y}_t^\xi \\ & + \lambda^\xi \left[1 - \frac{q^{-\xi}}{q^\xi} e^{-\gamma \hat{Y}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \right]\end{aligned}\tag{C.103}$$

In equilibrium, because there is a unit measure of symmetric investors, the aggregate asset holdings by investors $Y_t^\xi = \hat{Y}_t^\xi$. Market clearing requires that

$$Q_t^\xi + Y_t^\xi = S$$

where Q_t^ξ equals hedgers' aggregate asset holdings, and is given by (A.3). From (A.3) and (C.102),

$$\begin{aligned}& \left[\Sigma + \lambda^\xi (P_t^{-\xi} - P_t^\xi) (P_t^{-\xi} - P_t^\xi)^\top \right]^{-1} \left[\frac{\bar{D} - rP_t^\xi + \lambda^\xi (P_t^{-\xi} - P_t^\xi)}{\alpha} - \Sigma u^\xi \right] \\ & + \frac{1}{\gamma} \Sigma^{-1} (\bar{D} - rP_t^\xi) + \frac{\lambda^\xi}{\gamma} \frac{q^{-\xi}}{q^\xi} e^{-\gamma \hat{Y}_t^{\xi\top} (P_t^{-\xi} - P_t^\xi)} \Sigma^{-1} (P_t^{-\xi} - P_t^\xi) = S\end{aligned}\tag{C.104}$$

As explained earlier, the risky assets' prices depend only on the state of the economy ξ . Hence, we can define the state-dependent constant

$$\pi^\xi \equiv \frac{\bar{D}}{r} - P_t^\xi\tag{C.105}$$

Substituting (C.105) into (C.102), (C.103) and (C.104), and noting that $\hat{Y}_t^\xi = Y_t^\xi$, we have the following system of equations

$$Y_t^\xi = \frac{r}{\gamma} \Sigma^{-1} \pi^\xi + \frac{\lambda^\xi}{\gamma} \frac{q^{-\xi}}{q^\xi} e^{-\gamma Y_t^{\xi\top} (\pi^\xi - \pi^{-\xi})} \Sigma^{-1} (\pi^\xi - \pi^{-\xi})$$

and

$$-\rho = r(\log q^\xi + \log r - 1) + r \gamma Y_t^{\xi\top} \pi^\xi - \frac{1}{2} \gamma^2 Y_t^{\xi\top} \Sigma Y_t^\xi + \lambda^\xi \left[1 - \frac{q^{-\xi}}{q^\xi} e^{-\gamma Y_t^{\xi\top} (\pi^\xi - \pi^{-\xi})} \right]$$

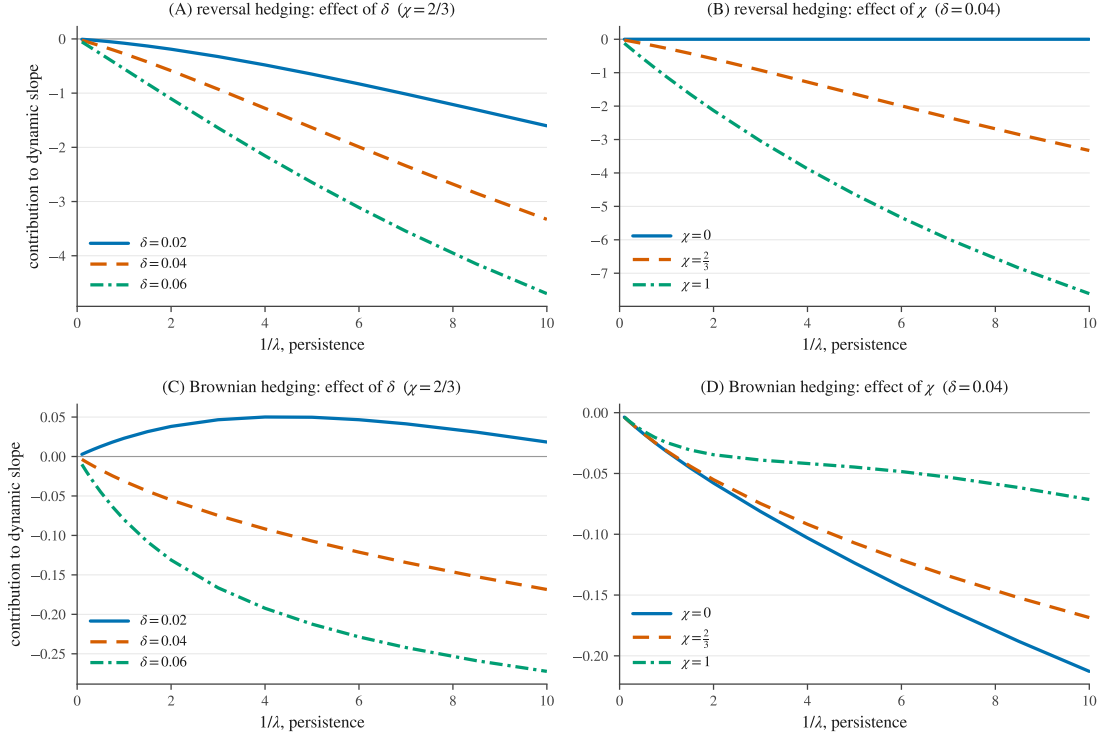


Figure D.1: Dynamic slope $\mathcal{C}$ and its decomposition for CRRA investors: myopic, Brownian hedging, and hedging against reversal components. We plot the decomposition for different size and persistence of the return shock. The parameter values are the same as in the baseline parameterization of Figure 2: $\bar{D} = 0.415$, $\sigma = 0.57$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S = 1$, $u^n = 0$, and all curves are evaluated at the mean of the stationary distribution of investors' wealth, $\bar{w} = 5.03$; we consider $\delta = 0.02, 0.04, 0.06$ and $\chi = 0, \frac{2}{3}, 1$.

as well as

$$\begin{aligned} & \left[\Sigma + \lambda^\xi (\pi^\xi - \pi^{-\xi})(\pi^\xi - \pi^{-\xi})^\top \right]^{-1} \left[\frac{r\pi^\xi + \lambda^\xi (\pi^\xi - \pi^{-\xi})}{\alpha} - \Sigma u^\xi \right] \\ & + \frac{r}{\gamma} \Sigma^{-1} \pi^\xi + \frac{\lambda^\xi}{\gamma} \frac{q^{-\xi}}{q^\xi} e^{-\gamma Y_t^{\xi\top} (\pi^\xi - \pi^{-\xi})} \Sigma^{-1} (\pi^\xi - \pi^{-\xi}) = S \end{aligned}$$

The equilibrium prices P_t^ξ of the risky assets at time t when the economy is in state ξ are

$$P_t^\xi = \frac{\bar{D}}{r} - \pi^\xi,$$

where $\xi \in \{n, s\}$.

D The Economics of the Myopic Slope vs. the Dynamic Slope

Figure D.1 explores the comparative statics of the three parameters (return size δ , persistence $1/\lambda$, and reversal share χ) for the two hedging components. Panel A and B of Figure D.1

plot the intertemporal hedging with respect to the (Poisson) reversal, which is relatively easier to explain. Investors understand that at some point the shift in residual supply will be reversed; when this occurs they realize a higher return, but, at the same time, their investment opportunity set gets worse. Investors hedge this by cutting their asset exposure, explaining the negative “reversal hedging.” Finally, it is intuitive that this negative effect gets stronger for larger shocks (larger δ) and larger reversal share (χ). To the extreme of $\chi = 0$ there is no hedging component due to reversal (Panel B).

Panel C and D then illustrate the hedging demand with respect to Brownian shocks, with risk aversion $\gamma > 1$.³⁶ From (40), the sign of Brownian hedging depends on the sign

$$\text{sgn} \left\{ \frac{q^h(w_t^n)}{q^h(w_t^n)} - \frac{q^n(w_t^n)}{q^n(w_t^n)} \right\} = \text{sgn} \left\{ \left(\frac{q^h(w_t^n)}{q^n(w_t^n)} \right)'_{w^n} \right\}. \quad (\text{D.106})$$

The simple expression in (D.106) captures the intuition of Brownian hedging. If the relative investment opportunities between two states $\frac{q^h(w_t^n)}{q^n(w_t^n)}$ decreases with the aggregate state w_t^n , then because the asset return is positively correlated with the Brownian shock, the investor should reduce her hedging demand of the asset with respect to (aggregate) Brownian shocks. Now, in line with Merton (1971) $q^n(w_t^n)$ is increasing in w_t^n for $\gamma > 1$, while the investment opportunity offered by the “deal” in Definition 2 features an extra expected return δ that is independent of w_t^n . To the extreme, when $\delta \rightarrow \infty$ the “deal” fully dominates the stochastic investment opportunity available in the economy and $q^h(w)$ becomes a constant. As a result, $\frac{q^h(w_t^n)}{q^n(w_t^n)}$ tends to be decreasing in w , giving rise to a negative Brownian hedging demand.

Several additional points are worth making on the comparative statics revealed in Figure D.1. First, it is intuitive that when the reversal share χ gets larger the Brownian component gets smaller. Second, when $\lambda \rightarrow \infty$, it is as if the investor received a fixed extra return δdt on her wealth over the interval $[t, t + dt]$, followed immediately by the equilibrium investment opportunity; hence $\frac{q^h(w)}{q^n(w)}$ should be a constant independent of w , which explains the zero hedging demand in all panels at the corner $1/\lambda \rightarrow 0$. Third, unlike the reversal component in Panel A, the Brownian component does not disappear as $\delta \rightarrow 0$. Its sign follows (D.106): under our calibration it is negative for larger shocks but positive for small ones (the $\delta = 0.02$ curve in Panel C), so that for small, persistent shocks the dynamic slope can lie slightly above the myopic one.

Finally, although in our calibrated applications the effect of the (two) dynamic hedging components is quantitatively small (see the small difference between the first two rows, the myopic and the dynamic slope, in both panels of Table 2), Figure D.1 shows that it can be large for a shock that is both large and persistent.

E Details of Calibration and Sensitivity Analysis

In this part, we give details on the calibration exercises which are the basis of Section 6.1. In our first calibration, we calibrate to the study of Ben-David et al. (2021) focusing on factor-level elasticities. In the second one, we are interested in an economy where the residual

³⁶It is well known since Merton (1971) that all else equal the sign of this component flips if $\gamma < 1$.

supply shock affects only very few assets out of many. Hence, we consider the case of index inclusion calibrating our model to the recent study of Greenwood and Sammon (2024).

E.1 Fund flows and the Factor-Elasticity

In this part, we calibrate our model to the Ben-David et al. (2021) study on factor-level elasticities. Then, using the obtained set of parameters as our baseline, we assess how our measures respond to a change of various parameters.

E.1.1 Baseline calibration

Our baseline parameters are obtained by calibrating our model to Ben-David et al. (2021). They show that when investment styles are highly correlated with Morningstar ratings, changes in fund ratings result in style-level fund flows which lead to price effects. In our calibration, we set $I = 2$. We interpret the two assets as two portfolios of different investment styles. In particular, we consider the first asset as the small value portfolio and the second asset as the large growth portfolio. The parameters $\bar{D}$ and σ pertain to the assets' fundamentals, and govern the first and second moments of portfolio returns. We set $\bar{D} = [0.27, 0.26]$ and $\sigma = \text{diag}(0.485, 0.485)$ to match the historical mean returns and return volatilities of the small value portfolio and the large growth portfolio as obtained from Ken French's data library.

We normalize the shocks in the normal state to zero, that is $u^n = [0, 0]$. We normalize the shock to the second asset to zero in the stressed state and set $u^s = [0.15, 0]$ such that the model-implied difference in flows between the two assets (interpreted as the top and the bottom styles) is about 13%, consistent with the reported style-based fund flows in Ben-David et al. (2021). We set $\lambda^n = 0$, that is the economy transitions from the normal state to the stressed state with intensity 0 and thus the normal state is the steady state in our calibration. Furthermore, we set $\lambda^s \equiv \lambda = 0.5$ (and hence a persistence of 2 of the residual supply shock) to match the empirical duration of fund flows as reported by Ben-David et al. (2021, Figure 5).

We set $r = 0.03$, which roughly equals the historical average risk-free rate. The investors' time discount rate ρ governs the aggregate asset holdings by hedgers. We interpret the hedgers in our model to include households, mutual funds and ETFs, foreign investors, etc. We set $\rho = 0.08$, which implies that hedgers hold 59% of the assets. Finally, we set $\gamma = 3$ which is taken from the literature. We normalize the hedgers' absolute risk aversion α and the supply of each asset to one. Our baseline parameters are provided in Table 1.

Using the baseline parameters, we simulate the distribution of investors' aggregate wealth. In the baseline, the mean wealth of investors is 2.11. In our model, the hedgers' endowment shocks give rise to trading needs by the hedgers. As the economy switches from the normal state to the stressed state, the hedgers wish to hold less of the assets leading to increased expected returns. At the mean investors' wealth level, our calibration implies an incremental return of $\Delta \mathbb{E}_t(R) = 3\%$, of which roughly 2/3 comes from the reversal.

It is worthwhile mapping our measured slope to the empirical demand elasticity à la Koijen and Yogo (2019), defined as the percentage of capitalization traded divided by the price impact. In Section E.2.2, we provide a detailed discussion of how our measured slope translates to demand elasticity. Using the baseline parameters, the corresponding demand

Incr. Return ($\Delta\mathbb{E}_t(R)$)	0.01	0.03	0.05	0.08	0.10
Myopic Slope	3.21	3.21	3.21	3.22	3.22
Dynamic Slope	3.20	3.17	3.14	3.07	3.01
Measured Slope	1.39	1.24	1.10	0.93	0.83
Measured Slope/Dynamic Slope	43%	39%	35%	30%	28%
Implied Price Elasticity	1.50	1.42	1.35	1.24	1.17
Dynamic minus Measured (DmM)	1.81	1.94	2.03	2.14	2.18
Shock to risk/DmM	114%	112%	110%	110%	111%
Diff Hedging Brownian/DmM	-15%	-13%	-12%	-10%	-10%
Diff Hedging Reversal/DmM	1%	1%	1%	0%	-1%

Table E.1: Comparative statics with respect to expected incremental return $\Delta\mathbb{E}_t(R)$, for the factor elasticity calibration. The colored (with bold font) column contains our benchmark parameterization: $\bar{D} = [0.27, 0.26]$, $\sigma = \text{diag}(0.485, 0.485)$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S = [1, 1]$, $u^n = [0, 0]$, $u^s = [0.15, 0]$, $\lambda = 0.5$, and $w = 2.11$. The remaining columns show how our effects vary with the expected incremental return.

Risk Aversion (γ)	0.7	1.5	2.0	3.0	3.5	log
Myopic Slope	9.59	5.37	4.34	3.21	2.84	7.28
Dynamic Slope	9.24	5.30	4.30	3.17	2.81	7.10
Measured Slope	1.43	1.48	1.41	1.24	1.15	1.49
Measured Slope/Dynamic Slope	16%	28%	33%	39%	41%	21%
Implied Price Elasticity	1.34	1.55	1.55	1.42	1.34	1.46
Dynamic minus Measured (DmM)	7.81	3.82	2.89	1.94	1.66	5.61
Shock to risk/DmM	97%	105%	109%	112%	112%	101%
Diff Hedging Brownian/DmM	4%	-6%	-9%	-13%	-13%	0%
Diff Hedging Reversal/DmM	-1%	0%	1%	1%	2%	-1%

Table E.2: Comparative statics with respect to risk aversion coefficient γ . The table shows comparative statics with respect to the investors' risk aversion coefficient γ , for the factor elasticity exercise. The colored (with bold font) column contains our benchmark parameterization: $\bar{D} = [0.27, 0.26]$, $\sigma = \text{diag}(0.485, 0.485)$, $\alpha = 1$, $r = 0.03$, $\rho = 0.08$, $S = [1, 1]$, $u^n = [0, 0]$, $u^s = [0.15, 0]$, $\lambda = 0.5$, and $w = 2.11$. The remaining columns show how our effects vary with the investors' risk aversion coefficient, with u^s recalibrated such that the implied expected return shocks remain constant at $\Delta\mathbb{E}_t(R) = 0.03$.

elasticity is 1.4. In Ben-David et al. (2021), the empirical factor elasticity is around 0.2. While our seven-fold difference is in line with the general observation that frictionless asset pricing models struggle to rationalize the empirically small factor elasticity, our difference is smaller than was perhaps previously thought.³⁷

³⁷While the exact numbers widely vary, Gabaix and Koijen (2022) suggest that the empirical estimates are several orders of magnitude smaller than their conceptual counterpart.

E.1.2 Results and slope analysis

Tables E.1-2 show the results of our baseline calibration and their sensitivity to various parameters. In each table the colored column contains our benchmark parameterization, while the other columns show how our effects vary. For each set of parameters, we calculate the myopic slope, the dynamic slope and the measured slope, and decompose the difference into our three effects. Implied elasticity is a scaled version of our measured slope to make it comparable to the estimates collected by Gabaix and Koijen (2022) from the empirical literature.

Table E.1 focuses on the non-linear effect of the size of the expected return shocks implied by the shift in residual supply. Table E.2 illustrates the effect of varying relative-risk-aversion of investors. Table 2 concentrates on the effect of the persistence of the demand shift.³⁸

There are a number of observations to make that are robust across our parameterizations.

First, consistent with the illustration in Figure 2, we find that dynamic slope is always smaller than the myopic slope. That is, Merton’s intertemporal hedging tends to make the dynamic demand curve flatter compared to its static equivalent. However, this effect is small.

Second, in each of our parameterizations, measured slope significantly undershoots its conceptual counterpart. In particular, measured slope is only 40% of the conceptual counterpart in our baseline calibration. This fraction shrinks to 16% when investors are less risk-averse than log. This illustrates the main observation of this paper. Because demand shifts change the slope of the demand curve in equilibrium, the econometrician relying on these shifts for estimating the demand slope is shooting a moving target. This calibration suggests that this results in significant underestimation of the targeted concept.

Table E.1 illustrates the non-linearity of this effect by varying the size of the shock. Note that the fraction of measured slope to dynamic slope is shrinking when shocks are larger implying larger mismeasurement.

Third, the endogenous risk effect is responsible for the largest part of this mismeasurement. In fact, because the amplified intertemporal hedging component is going against the endogenous risk effect whenever $\gamma > 1$, in all these cases the endogenous risk effect is more than 100% of the total under-measurement. This share is growing with risk-aversion and investor’s wealth and shrinking with the persistence of the shock ranging from 97% (in our only parameterization with $\gamma < 1$) to 134%.

Finally, while the GE amplified intertemporal hedging component has the same sign as Merton’s intertemporal hedging effect, it is several orders of magnitude higher. In absolute terms, it can reach 34% of the total mismeasurement. Its absolute magnitude is larger for less persistent shocks, larger wealth and for greater risk-aversion.

E.2 Index Inclusion: Micro-Elasticity

In our second application, we are interested in an economy where the residual supply shock affects only very few assets. To make this point clear, we picture a market with a large number of assets with the same cash-flow risk. That is, all I assets have identical average dividend $\bar{D}$, and $[\Sigma]_{ii} = \sigma^2$ and $[\Sigma]_{ij} = \kappa\sigma^2$ and their supply is $[s]_i = s$ for all i and $j \neq i$. The only difference across these assets is in hedgers’ endowment. In particular, suppose that

³⁸In Tables E.2-2, for each column we recalibrate the parameters u^ξ to ensure that the implied expected return shocks remain constant at $\Delta\mathbb{E}_t(R) = 0.03$.

in the normal state, hedgers' endowment for the first I_1 assets differs from the rest:

$$[u^n]_{i=1}^{I_1} = v_1, [u^n]_{i=I_1+1}^I = v_2.$$

where $v_1 \neq v_2$ are scalars.

Our interpretation is that the first I_1 assets are in an index. We are interested in a shock for which $[u^n]_{i=2}^{I-1} = [u^s]_{i=2}^{I-1}$ while $[u^s]_1 = v_2$, $[u^s]_I = v_1$. That is, the first asset is excluded from and the last asset is included in the index.

Note that we intentionally design this example, so that the shock itself has a minimal effect on the structure of the economy—effectively, we only swap the labels of the first and the last assets. Hence, one might expect that the shock should have a negligible effect on the equilibrium outcome; hence, the informativeness of shifts to residual supply should perhaps be restored. We show that this is not the case.

In this part, we calibrate to Greenwood and Sammon (2024), who study the measured slope of index inclusion and deletion events. We set $I = 502$ and $I_1 = 501$ such that the index in our calibration corresponds to the S&P 500 Index. Asset 1 and asset 502 switch in and out of the S&P 500 Index, while assets 2 to 501 are always in the index. $\bar{D}_i$ is the expected cash flow of asset i , where $i \in \{1, 2, \dots, I\}$. σ_i is the asset's cash flow volatility. κ is the pairwise cash flow correlation between asset i and asset j , where $i \neq j \in \{1, 2, \dots, I\}$. We calibrate $\bar{D}_i$ and σ_i to match historical return moments. We set $\kappa = 0.3$ such that roughly 75% of an asset's return volatility is idiosyncratic (see Campbell, Lettau, Malkiel and Xi (2001)).

As in our calibration of the factor-elasticity, we normalize the hedgers' absolute risk aversion α to one. We set $r = 0.03$, which roughly equals the historical risk-free rate. We also set the investors' time discount rate $\rho = 0.08$, the same as in the factor-elasticity calibration. ρ affects the aggregate asset holdings by hedgers, which are interpreted as mutual funds and ETFs, pension funds, etc. In our calibration, hedgers hold 45% of index assets and 2% of non-index assets. We set $\gamma = 3$, which is taken from the literature.

We normalize the supply of each asset to $1/\sqrt{I} = 0.0446$. To match the fund flows due to index inclusion and deletion in Greenwood and Sammon (2024), we normalize hedgers' endowment shock v_1 to zero for index assets, and set their endowment shock for non-index assets $v_2 = 0.0107$. These give us a fund flow of roughly 13% which is in-line with industry research cited by Greenwood and Sammon (2024). Finally, we set the intensity of state transitions $\lambda = 0.2$, which targets the conditional probability of index inclusion or deletion. λ is computed as the number of stocks included (or excluded) in a given year, divided by the number of stocks within the band on watch for inclusion (or exclusion). We conduct sensitivity analysis with respect to λ and γ .

Using the baseline parameters, we simulate the distribution of investors' aggregate wealth. In the baseline, the mean wealth of investors is 4.25. In our model, the index inclusion/deletion changes hedgers' relative endowment shock across the affected assets, giving rise to trading needs by the hedgers. As expected for such a local shock, the implied mean incremental return is very small, $\Delta \mathbb{E}_t(R_1) \approx 0.01\%$. One of the main points of this calibration is that even if identification relies on a demand shock implying tiny effects on expected returns, the implied mismeasurement of demand slope due to general equilibrium effects is not diminishing.

Risk Aversion (γ)	0.9	1.5	2.0	3.0	3.5	log
Myopic Slope	30.41	17.58	12.54	7.85	6.55	27.29
Dynamic Slope	30.41	17.58	12.54	7.85	6.55	27.29
Measured Slope	7.87	6.01	4.81	3.44	2.99	7.54
Measured Slope/Dynamic Slope	26%	34%	38%	44%	46%	28%
Implied Price Elasticity	672	569	500	399	357	648
Dynamic minus Measured (DmM)	22.49	11.58	7.73	4.41	3.55	19.75
Shock to risk/DmM	98%	107%	111%	116%	118%	100%
Diff Hedging Brownian/DmM	2%	-7%	-11%	-16%	-18%	0%
Diff Hedging Reversal/DmM	0%	0%	0%	0%	0%	0%

Table E.3: Comparative statics with respect to risk aversion coefficient γ . The table shows comparative statics with respect to the investors' risk aversion coefficient, for the micro elasticity calibration. The colored (with bold font) column contains our benchmark parameterization: $\bar{D}_i = 0.0369$, $\sigma_i = 0.0692$, $\kappa = 0.3$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S_i = 0.0446$, $v_1 = 0$, $v_2 = 0.0107$, $\lambda = 0.2$ and $w = 4.25$. The remaining columns show how our effects vary with the risk aversion coefficient, with v_2 recalibrated such that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.01\%$.

E.2.1 Results and slope analysis

Tables E.3-E.4 show the results of our baseline calibration and their sensitivity to various parameters. As before, in each table the colored column contains our benchmark parameterization, while the other columns show how our effects vary.

Table E.3 illustrates the effect of varying relative-risk-aversion of investors. Table 2 concentrates on the effect of the persistence of the demand shift. Table E.4 highlights the effect of state dependence: how our measures change when investors' wealth is below and above the mean.³⁹

The results are remarkably similar to our factor-elasticity results even in a quantitative sense. Dynamic Slope is somewhat smaller than myopic slope. Measured slope is a fraction of its theoretical counterpart (amounting only to 26% when risk-aversion is small). The bulk of the effect is due to the endogenous change in the covariance matrix, but the GE amplified intertemporal effect remains significant.

In this calibration, we also present how our effects depend on the state of the economy, i.e. the level of aggregate wealth of investors, w_t in Table E.4. This illustrates that all our objects of interest, myopic slope, dynamic slope and measured slope are highly state dependent. Measured slope is only 44% of demand slope in our baseline calibration for the micro elasticity calibration exercise. As the investor wealth increases from 4.25 to 4.75, this fraction also increases reaching 57%. The corresponding implied price elasticity is monotonically increasing in investor wealth, suggesting that a lower wealth share of the investor sector decreases the implied price elasticity measured by econometricians. In practice, many investors are often "sleepy" and do not engage in active trading. This reduces the effective investor wealth, causing the implied price elasticity to be smaller.

It is also useful to observe that the hedging against the reversal effect is even smaller in

³⁹In Tables E.3, E.4 and 2, for each column we recalibrate the parameters u^ξ to ensure that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.01\%$.

Investor Wealth (w)	1.00	2.50	3.50	4.25	4.75
Myopic Slope	3.21	5.12	6.68	7.85	8.55
Dynamic Slope	3.21	5.12	6.68	7.85	8.55
Measured Slope	1.49	2.08	2.66	3.44	4.90
Measured Slope/Dynamic Slope	46%	41%	40%	44%	57%
Implied Price Elasticity	79	218	322	399	442
Dynamic minus Measured (DmM)	1.72	3.04	4.02	4.41	3.65
Shock to risk/DmM	109%	114%	115%	116%	118%
Diff Hedging Brownian/DmM	-9%	-14%	-15%	-16%	-18%
Diff Hedging Reversal/DmM	0%	0%	0%	0%	0%

Table E.4: Comparative statics with respect to investor wealth w . The table shows comparative statics with respect to the investors' aggregate wealth w , for the micro elasticity calibration. The colored (with bold font) column contains our benchmark parameterization: $\bar{D}_i = 0.0369$, $\sigma_i = 0.0692$, $\kappa = 0.3$, $\alpha = 1$, $\gamma = 3$, $r = 0.03$, $\rho = 0.08$, $S_i = 0.0446$, $v_1 = 0$, $v_2 = 0.0107$, $\lambda = 0.2$ and $w = 4.25$. The remaining columns show how our effects vary with investor wealth, with v_2 recalibrated such that the implied expected return shocks remain constant at $\Delta \mathbb{E}_t(R) = 0.01\%$.

the index inclusion exercise than in our factor-elasticity calibration. It is so, because this is the only one which diminishes with the size of the shock.

Under the baseline parameters, the index-inclusion calibration yields an implied price elasticity à la Kojien and Yogo (2019) of 399, which is substantially lower than the commonly cited value of around 6250 from Petajisto (2009). For comparison, the empirical micro elasticity estimated by Greenwood and Sammon (2024) is approximately 2.7.

E.2.2 Translating Measured Slope to Elasticity

As explained earlier, the demand elasticity is defined as the percentage of capitalization traded divided by the percentage change in price, that is

$$\mathcal{E}(w_t; \lambda, u^s) = \frac{\frac{w^s \hat{y}_i^s}{S_i P_i^s} - \frac{w^n \hat{y}_i^n}{S_i P_i^n}}{-\frac{P_i^s - P_i^n}{P_i^n}}, \quad (\text{E.107})$$

where we have suppressed the time index for simplicity. $\hat{y}_i^n$ and P_i^n are the investor's portfolio share and the equilibrium price of asset i in the normal state. $\hat{y}_i^s$ and P_i^s are portfolio share and price in the stressed state. The investor has wealth w^n in the normal state and w^s in the stressed state. Notice that (E.107) can be rewritten as

$$\mathcal{E}(w_t; \lambda, u^s) = \frac{1}{y_i^n} \frac{[w^n \text{diag}(P^n)^{-1} y^n]_i}{S_i} \frac{\frac{P_i^n w^s}{P_i^s w^n} \hat{y}_i^s - \hat{y}_i^n}{\frac{P_i^n - P_i^s}{P_i^n}}.$$

The middle term on the right-hand side is the fraction of asset i 's capitalization held by investors. The last term on the right-hand side is of the same order of magnitude as the measured slope. The equation thus suggests that the demand elasticity is inversely related to investors' portfolio share of asset i .

Para.	Description	Value	Targeted Moments
$\bar{D}$	Expected cash flow of assets	0.415	historical returns
σ	Cash flow volatility	0.57	historical return volatilities
α	Hedgers' absolute risk aversion	1	normalization
r	Risk-free rate	0.03	historical average interest rate
ρ	Investors' time discount rate	0.08	investor holdings
S	Asset supply	1	normalization
u^n	Hedgers' endowment in normal state	0	normalization
u^s	Hedgers' endowment in stressed state	0.01	fund flows Gabaix and Koijen (2022)
λ	Intensity of state transitions	1	persistence Gabaix and Koijen (2022)

Table E.5: Parameters and Calibration Details for the Single Asset, Log Investors Case. The table reports the parameter values used in the single asset, log investors calibration exercise.

The implied demand elasticity $\mathcal{E}(w_t; \lambda, u^s)$ depends on the shock persistence $1/\lambda$. For our comparative statics, we fix the incremental expected return $\Delta \mathbb{E}_t(R) \equiv \mu_{Rt}^s - \mu_{Rt}^n + \lambda \text{diag}(P_t^s)^{-1}(P_t^n - P_t^s)$ at the level implied in the baseline. Fixing $\Delta \mathbb{E}_t(R)$, the change in asset prices $\text{diag}(P_t^s)^{-1}(P_t^n - P_t^s)$ is higher when λ is low, that is when the shock is more persistent. In other words, transitory shocks tend to have smaller price impact. Consequently, when $1/\lambda$ is small, the denominator in (E.107) is also small, magnifying the implied demand elasticity. Notice that our measured slope $\mathcal{M}(w_t; \lambda, u^s)$ adjusts for this horizon effect due to shock persistence.

E.3 Single Asset, Log Investors Calibration

In Section 4.2, we examine a simple case with a single asset and log investors. The baseline parameter values are shown in Table E.5.

For the single asset, log investors case, we calibrate to Gabaix and Koijen (2022), which studies the macro elasticity. Corresponding to their framework, the single asset in our calibration can be interpreted as the stock market. $\bar{D}$ is the expected cash flow of the stock market, and σ is the cash flow volatility. We calibrate $\bar{D}$ and σ to match historical return moments. Throughout this exercise, model-implied return moments are computed as averages under the stationary distribution of investors' wealth, while the slope measures and their decomposition are evaluated at the mean of this distribution, $\bar{w} = 3.24$. The calibrated average return is 8.8%, in line with its historical counterpart of 8.9%, while the average return volatility is 19.1%, above its historical counterpart of 16.4%; the gap reflects that log investors are not sufficiently risk averse to match both moments simultaneously. With $\gamma = 3$, the specification used in Figure 2, the same cash-flow parameters imply an average return of 7.8% and an average return volatility of 14.9%, with a stationary mean wealth of 5.03.

Moreover, as in our calibration of the factor-elasticity and micro-elasticity, we normalize the hedgers' absolute risk aversion α to one. We set $r = 0.03$, which roughly equals the historical risk-free rate, and set $\rho = 0.08$ to be consistent with the factor-elasticity and micro-elasticity calibrations. We further normalize the asset supply to one and u^n (hedgers' endowment in the normal state) to zero.

Finally, the endowment in the stressed state $u^s = 0.01$ is chosen such that the implied

fund flow is roughly 1%, as documented by Gabaix and Koijen (2022). In Figure 1, we plot the demand slope of a log investor, $\mathcal{C}^{\log}$ and the corresponding measured slope $\mathcal{M}^{\log}|_{\delta=\Delta\mathbb{E}_t(R)}$ against the implied change in percentage incremental expected return $\Delta\mathbb{E}_t(R)$ by varying the u^s shock. The intensity parameter λ matches the inertia parameter in Gabaix and Koijen (2022) who also consider a case of short-term vs. long-term elasticity when funds are inertial. The half-life in their calibration is 0.7, which translates into $\lambda = 1$ in our model. In Figure 1, we consider two limit cases with $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ respectively.

The implied price elasticity of the single asset, log investors calibration is 2, compared to the macro elasticity estimate of 0.2 in Gabaix and Koijen (2022).