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A Theory of Debt Maturity: The Long and Short of Debt Overhang

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ABSTRACT

Debt maturity influences debt overhang, the reduced incentive for highly levered borrowers to make real investments because some value accrues to debt. Reducing maturity can increase or decrease overhang even when shorter term debt's value depends less on firm value. Future overhang is more volatile for shorter term debt, making future investment incentives volatile and influencing immediate investment incentives. With immediate investment, shorter term debt typically imposes lower overhang; longer term debt can impose less if asset volatility is higher in bad times. For future investments, reduced correlation between assets-in-place and investment opportunities increases the shorter term debt overhang.

THIS PAPER STUDIES THE effects of debt maturity on the current and future real investment decisions of an equity owner (or a manager who is compensated by equity). Our analysis is based on debt overhang first analyzed by Myers (1977), who points out that outstanding debt may distort the firm's investment incentives downward. Reduced incentives to undertake profitable investments when decision makers seek to maximize equity value is referred to as "debt overhang," because part of the return from a current new investment makes existing debt more valuable.

Myers (1977) suggests short-term debt as a possible solution to the debt overhang problem. In part, this extends the idea that, if all debt matures before the investment opportunity, then the firm without debt in place can make investment decisions as if an all-equity firm. Following this logic, debt that matures soon—although *after* relevant investment decisions, as opposed to *before*—should have reduced overhang.

However, short-term debt is known to have several disadvantages. For firms without access to outside funds to meet debt repayments, short-term debt can

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lead to early firm closure and liquidation (e.g., Diamond (1991), Gertner and Scharfstein (1991)). More relevant to our paper, Gertner and Scharfstein (1991) show that, conditional on ex post financial distress, making a fixed promised debt payment due earlier (i.e., shorter term) raises the market value of the debt and thus the firm's market leverage, leading to more debt overhang ex post. In addition, certain drawbacks of short-term debt have been suggested by some quantitative models (Titman and Tsyplakov (2007), Moyan (2007)) that focus on equity holders' differential abilities and incentives to adjust leverage in response to new information given different debt maturity structures.[1]

Our paper aims to provide a thorough analysis of the effects of debt maturity on equity incentives to undertake both current and future investments, and, more importantly, to identify the forces that determine overhang. We show why the ideas based on Myer's suggestion have merit, and show how and why they can be reversed under different settings. Throughout, we first illustrate our results via simple examples; these results are then generalized in the context of standard models used by practitioners and researchers, allowing us to establish their generality and introduce more complicated issues.

We stress the importance of the relative timing of the dates associated with investment decisions, the maturity of debt, and the arrival of news about the prospects of the firm's existing assets (including past investment). In a nutshell, the value of shorter term debt is less sensitive to the value of the firm and thus would seem to receive a smaller benefit from new investment taken just after the debt is issued, which is equivalent to lower overhang than long-term debt. This is consistent with the ideas in Myers (1977). However, for future investment opportunities, future prospects and the value of existing assets will fluctuate before these investment decisions are made. The lower sensitivity to firm value of shorter term debt implies a more volatile equity value and hence a more volatile future state-contingent debt overhang. Shorter term debt thus imposes stronger overhang in bad times, a result related to those in Gertner and Scharfstein (1991) and Titman and Tsyplakov (2007).

The first setting that we analyze is one in which an investment decision is made right after the issuance of the debt but before any new information is released about the value of existing assets. We present a three-date example where debt is issued at the initial date with an investment decision made immediately thereafter. We compare short-term debt, which matures on the middle date, with longer term debt, which matures on the final date. This example extends the logic of Myers (1977), where shorter term debt imposes less overhang. In the example, a shorter maturity makes the market value of debt less sensitive to changes in firm value, and equity holders' investment incentives are distorted (downward) by the spillover of the increased value of claims other than equity. However, we present two additional examples to show that this intuitive idea is incomplete, and illustrate how this effect of maturity could reverse. The first example shows the importance of investment decision timing in a three-date example where the investment is made after

[1] For more details, see the literature review at the end of the introduction.

some resolution of uncertainty about existing assets but before even short-term debt has matured. The second example returns to the case in which the investment decision is made immediately after debt issuance and shows how the effect of debt maturity on overhang depends on the way that uncertainty is resolved over time. More specifically, this example illustrates the effect of state-dependent volatility of assets.

Our first example with constant asset volatility and an immediate investment decision taken before any resolution of uncertainty illustrates the basis for the possibility of lower overhang for shorter term debt *with the same market value as longer term debt*. We prove this result for an immediate investment in the model of Black and Scholes (1973) and Merton (1974) in Proposition 1 of Section II. To our knowledge, this is a new analytical result, because as maturity varies, our result holds constant the debt value and hence the firm's leverage on the date of issuance. The classic analysis of the effect of varying maturity on the risk of a firm's bonds in Merton (1974) holds constant the promised payment (the face value for zero-coupon bonds), not the market value of bonds of differing maturities. To understand the effect of maturity on debt overhang, it is important to hold the market value of debt constant by varying the promised payments. Holding constant the borrower's leverage in this way and thus focusing exclusively on debt maturity has not been stressed in previous studies on the effect of maturity on debt overhang, for example, Gertner and Scharfstein (1991).

In the Black-Scholes-Merton setting, asset volatility is constant. If the volatility of the value of assets-in-place is instead sufficiently higher after bad outcomes than good ones, shorter term debt may have stronger overhang even for immediate investment decisions. We show this in an example in Section I.D and more formally in Section II.C by adding state-dependent volatility to the Black-Scholes-Merton model. These results produce new implications for the effect of debt maturity on immediate investment incentives.

In the second setting that we analyze, firms have many investment opportunities in the present and the future. With future investment opportunities, the distribution of debt overhang in the future is relevant because the incentive to invest depends on the overhang prevailing at the time of future investment. In addition, debt overhang influences equity's decision to default on debt,[2] and default implies that future investment opportunities are not taken. In general, exactly because shorter term debt is less sensitive to changes in firm value, it leads to more volatile future equity value and hence more volatile future overhang: equity has weak investment incentives and is more likely to default after poor performance of the firm's assets-in-place, and has strong investment incentives after good performance. Examples in Section I.D illustrate this logic and some of its implications. For firms with investment opportunities in the future, a balance of maintaining investment incentives in future good and

[2] The interpretation of endogenous default given debt burden as "underinvestment" due to debt overhang is mentioned in, for example, Lambrecht and Myers (2008) and He (2011).

bad states leads to an optimal interior maturity structure. This mechanism is formally illustrated in Section III, where we study a dynamic model that generalizes Leland (1994b, 1998) to include a series of future investment opportunities.

We further show that, when investment opportunities are positively correlated with the firm's assets-in-place, it is beneficial to have a shorter debt maturity, which imposes lower (higher) debt overhang in times with a high (low) value of assets-in-place and hence better (worse) investment opportunities. This result offers a new perspective on empirical predictions regarding growth firms and debt maturity. For growth firms with uncertain investment opportunities where existing investment projects have realizations that are positively correlated with the value of new investment opportunities, shorter term debt is preferred. However, for growth firms with known future opportunities or for which realized asset returns are not very informative about future opportunities, investment incentives are more efficient with longer term debt. At the other extreme, mature firms that require timely maintenance to replace unexpectedly high depreciation in times of low cash flows should choose even longer term debt. This perspective is different from the existing idea that firms with substantial future investment opportunities should choose shorter term debt.

In our dynamic setting with multiple investment opportunities, today's investment benefit is positively related to future investment policies. Because shorter term debt triggers earlier default, which eliminates future growth opportunities, this negative force may feed back to today and undermine current investment incentives. Intuitively, if future growth extending today's investment will not occur, current investment is less attractive. This logic has interesting implications in our dynamic setting. In contrast to a static setting where riskless debt cannot impose any overhang, Section III.D shows that in our dynamic setting a policy of (almost) riskless ultrashort debt may cause strong overhang on current investment.

Our framework focuses on investment decisions during the period before debt refinancing and thus fits well with the empirical literature in which short-term debt is often classified as that with maturity of three years or less (e.g., Johnson (2003)). Indeed, empirical work on debt maturity based on the hypothesis of reduced overhang of shorter term debt, which implies the use of more short-term debt by "growth" firms with large investment opportunities, has had mixed success for reasons related to our findings in this paper.[3]

Short-term overhang is related to, but distinct from, the idea that shorter term debt maturity increases the control rights of lenders to discipline management (e.g., Calomiris and Kahn (1991), Diamond (1991), Flannery (1994),

[3] For instance, Barclay and Smith (1995) and Guedes and Opler (1996) document a negative relation between maturity and growth opportunities, while Stohs and Mauer (1996) and Johnson (2003) find a positive relation after controlling for firm leverage.

Leland (1998), Diamond and Rajan (2001a), and Benmelech (2006)).[4] Short-term debt provides discipline in part because short-term lenders do not sit idly while a borrower misbehaves—they demand payment on maturity. This is closely related to our point that short-term debt can have severe overhang when firm value declines after the debt is issued. From this perspective, our paper extends the results in Gertner and Scharfstein (1991) that for a given total promised (but risky) payment to debt holders, ex post debt overhang is made worse if more of the fixed amount is due sooner. Making a fixed payment due earlier raises the market value of the debt and thus the firm's market leverage, increasing ex post overhang. Our paper emphasizes the timing of investment, and examines ex ante effects of maturity on debt overhang for a *given* initial leverage to isolate it from the pure leverage effect.[5] Hence, our paper reconciles the result in Gertner and Scharfstein (1991) that shorter term debt has more ex post overhang with the suggestion in Myers (1977) that shorter term debt has less ex ante overhang.

We describe several important effects of debt maturity that have not received much formal analysis in the literature. We provide new results on the effect of debt maturity on debt overhang with an immediate investment in the Black-Scholes-Merton setting, holding leverage constant. Our result on the effects of higher volatility when firm value is low has not been suggested previously. We argue in the conclusion that this result applies especially well to financial institutions such as banks. Our analytical study for the case of dynamic investment opportunities in the Leland setting is related to existing quantitative studies on the effect of debt overhang. These studies focus on leverage adjustments to trade off the tax shield against physical costs of default, and each adds some other frictions. They all suggest that short-term debt is not a perfect solution to overhang. Titman and Tsyplakov (2007) use a model based on Leland (1998) but with costs of adjusting leverage, and find that short-term debt improves investment incentives but triggers earlier default. Relative to Titman and Tsyplakov, our paper with a simpler setting provides analytical results and further points out that, with intertemporally linked investment incentives, short-term debt may hurt current investment decisions due to earlier default in the future. In another closely related paper, Moyan (2007) studies the effect of debt maturity on overhang directly but focuses on an assumed asymmetry in leverage adjustment, that is, leverage cannot be adjusted if there is long-term debt but can be adjusted every period if short-term debt is issued. Moyan finds that, compared to long-term debt, a firm with short-term debt has higher (lower)

[4] It also differs from theories of maturity structure where short-maturity debt is attractive to borrowers with private information that their future credit rating may improve (e.g., Flannery (1986) and Diamond (1991)), or the idea that short-term debt entices runs due to coordination issues (e.g., Diamond and Dybvig (1983), He and Xiong (2012b)).

[5] We study the effect of maturity on debt overhang by fixing leverage exogenously for an additional reason. Tax-based theories suggest that leverage increases when current and future investment opportunities become more profitable, while control and pecking order theories suggest the opposite. There is no consensus in empirical studies about these determinants of dynamic adjustments to firm leverage (e.g., Fama and French (2002)).

leverage in good (bad) times, but the overall overhang effect is similar across both maturity structures. In contrast, we follow Leland (1994b, 1998) where the debt burden is fixed to focus only on maturity.

Our analysis of dynamic investment opportunities is based on Leland (1994b, 1998) so that the debt refinancing rate (which is inversely related to the firm's debt maturity structure) is fixed at a constant. The dynamic adjustment of debt maturity is beyond the scope of our paper, but we provide some discussion in Section III.F.

The rest of the paper is organized as follows: We give a series of examples in Section I to illustrate the key ideas of this paper. Section II provides a model with a single investment decision based on the Black-Scholes-Merton setting, and Section III provides a dynamic Leland-type model with many future investment decisions. In Section IV we conclude.

I. Debt Overhang: Assumptions and Examples

We first describe debt overhang and related assumptions, and then provide numerical examples to illustrate the main insights delivered by the paper.

A. Debt Overhang and Key Assumptions

Debt overhang, first formalized by Myers (1977), captures the insight that investment often leads to external benefits that accrue to the firm's debt claims. These external benefits consequently lead equity holders (or equivalently, managers who are paid in equity) who make investment decisions to internalize only part of investment benefits, and hence to underinvest relative to the level that maximizes the total value of the firm.

To study debt overhang, we make the following assumptions throughout:

(1) We examine standard debt contracts with two characteristics: promised face value and maturity.
(2) We assume that at date 0 the firm has to raise a certain amount of financing through debt. Our analysis fixes the initial market value of the debt, because we study debt maturity for a given amount of leverage. However, we do not specify the particular reasons why firms use debt. This is in part because all reasons for using debt must take account of the potential effect on investment incentives, and in part because there is no empirical consensus on the relative merits of various reasons (e.g., tax or managerial incentives for decisions other than investment; see footnote 5).
(3) We assume that it is equity holders who control the firm and who carry out investment. This assumption captures the idea that corporate decisions are delegated to those in control, rather than decided by a consensus of outside investors. Investment opportunities are lost during bankruptcy, and we impose no exogenous bankruptcy cost otherwise.

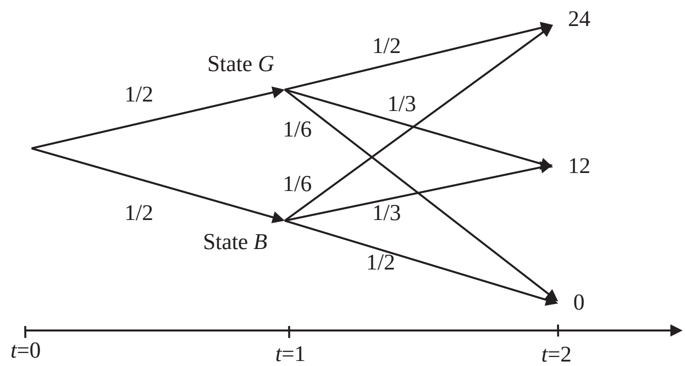

Figure 1. Timeline for numerical examples, with conditional probabilities denoted on each path. As shown, the conditional distribution given state G is {1/2, 1/3, 1/6}. We will also consider an alternative conditional distribution of {1/3, 2/3, 0} given state G.

(4) We assume that debt cannot be renegotiated to bribe managers to make alternative decisions. This assumption is especially relevant to debt with many holders, as opposed to a single bank or individual.[6]

(5) We focus on investment projects that are subject to debt overhang only, that is, projects that weakly increase or leave unchanged the value of each of its debt and equity claims. We do not consider "risk shifting," where a large increase in the risk profile of existing assets may cause a redistribution of value across equity and debt claims, as described in Jensen and Meckling (1976). Throughout this paper we focus on incremental investments that have less chance of introducing the possibility of risk shifting.

(6) To focus on maturity only, debts with different maturities are assumed to have the same seniority during bankruptcy.[7]

B. Example Setting

We begin by showing our results via numerical examples. We later show similar results based on standard Black-Scholes-Merton and Leland models.

Assets-in-place: As in Figure 1, the firm has assets-in-place that bring final cash flows at date 2 with three potential outcomes {24, 12, 0}, each occurring with probability 1/3 from the perspective of date 0. There are no cash flows on other dates. The discount rate is zero.

[6] Gertner and Scharfstein (1991) show that debt that cannot be renegotiated (especially if short-term) can impede renegotiation of other debt. In that sense, even renegotiation is subject to overhang.

[7] This is related to the dilution effect when firms refinance their maturing debt in a dynamic model. We rule out dilution by adopting the Leland (1994b, 1998) setting in Section III. For dilution issues, see Diamond (1993), Hackbarth and Mauer (2012), Dangl and Zechner (2008), Brunnermeier and Oehmke (2013), and related discussion in Section III.F.1.

Information: Just before date 1 some public information arrives. At state B, which occurs with probability 1/2, the news is bad and the conditional probabilities to reach the final outcomes become {1/6,1/3,1/2}. Symmetrically, good news arrives at state G with a probability of 1/2, and the conditional probabilities for the three outcomes become {1/2,1/3,1/6}.

Debt face values and maturities: Suppose that the firm needs to raise 8.25 at date 0. The debt can be either long-term (repaid at date 2) or short-term (repaid at date 1), with a face value of $F_L = 12.75$ or $F_S = 8.5$, respectively, leading to the target date-0 market value of 8.25:

$$\frac{1}{3} \times 12.75 + \frac{1}{3} \times 12 + \frac{1}{3} \times 0 = 8.25 = \frac{1}{2} \times 8.5 + \frac{1}{2} \times 8. \tag{1}$$

The left-hand side of equation (1) describes the payoffs to long-term debt, which is only paid in full (i.e., 12.75) with probability 1/3 at the outcome of 24. The right-hand side describes the short-term debt: with probability 1/2, the firm in state G pays debt holders the full face value 8.5, while with probability 1/2 the firm in state B defaults, and short-term debt holders recover the assets-in-place with a value of $8 = 24(1/6) + 12(1/3)$.

Investment opportunities: For ease of illustration we only consider *infinitesimal* investment, which improves the final payoff of assets-in-place by $\varepsilon > 0$. We do not specify the investment cost because debt overhang can be measured by the investment benefit that is captured by debt. The investment decision will be made only if its net present value (NPV) exceeds the debt overhang.[8]

Investment timing: We consider two different investment timings. First, the firm invests just after date 0 before the realization of state G or B, but after the debt is issued; second, the firm invests just before date 1 after the realization of the news about the assets-in-place (state G or B) but before the short-term debt matures. We believe both timing assumptions are empirically relevant.

C. Date-0 Investment before News about Assets-in-Place: A Benchmark Result

We first consider the case of a single investment just after date 0, immediately after raising the debt. Many of the existing ideas based on the discussion in Myers (1977) consider the effect of maturity on debt overhang in this particular setting.

We calculate the overhang as the expected benefit from the new investment that is captured by the debt with given maturities. Because the long-term debt face value $F_L = 12.75$ exceeds the intermediate outcome 12 but is below the highest outcome 24, the overhang occurs in both the middle and low states and thus is $2/3\varepsilon$ (equity gets $1/3\varepsilon$). For short-term debt, in state B the firm value 8 is below the face value of short-term debt $F_S = 8.5$. Short-term debt imposes an overhang of $1/2\varepsilon$ as it captures all of the gain at state B from the

[8] We assume that the investment adds a constant amount to each payoff, but essentially similar results hold if the investment increases each payoff by a multiplicative factor greater than one. The Black-Scholes-Merton model in Section II studies such multiplicative scale expansions.

date-0 investment. If the investment cost at date 0 is between $1/3\varepsilon$ and $1/2\varepsilon$, then this investment will be taken if and only if the firm uses short-term debt. In this example, long-term debt imposes more overhang than short-term debt, consistent with the discussion of Myers (1977). We formally show this result in Section II using a model based on Black-Scholes (1973) and Merton (1974).

This intuitive result relies upon two assumptions that we study in the following subsections. The first is about investment timing; we show that, if equity holders make an investment decision at date 1 after the news about assets-in-place, then the optimal maturity will depend on the details of the investment opportunities. The second is about the cash-flow distribution; we show that, even for date-0 investment, short-term debt may impose stronger overhang in a distribution featuring higher volatility following bad news.

D. Future Investment: Date-1 Investment after News about Assets-in-Place

Now suppose that investment opportunities are available only just before date 1. This implies that equity holders make investment decisions after the realization of the interim state but before the short-term debt matures. Consider the case of long-term debt first. At state G, the benefit from an infinitesimal investment that goes to debt holders is $1/2\varepsilon$ (equity also gets $1/2\varepsilon$); put differently, equity recovers the benefit from investment only at the outcome 24, which occurs with a conditional probability of 1/2. A similar argument implies that at state B the long-term debt overhang is $5/6\varepsilon$ (equity gets $1/6\varepsilon$). Hence, long-term debt imposes some overhang in both states, but it is never so severe that equity holders recover nothing from new investment. If the cost of investment is less than $1/6\varepsilon$, for example, then there will be investment in both states.

In contrast, short-term debt is a hard contract that does not share as much risk with equity due to its requirement of full payment whenever possible on its short maturity. As a result, in state G short-term debt imposes no overhang, but in state B it imposes the most extreme overhang, which is ε so that short-term debt holders capture the entire benefit of investment. To see this, the short-term debt becomes riskless at state G (the firm value 16 exceeds the debt face value 8.5) and therefore will not capture any gain from new investment. However, at state B, the deteriorating assets-in-place with a value of 8 fall below the face value 8.5, so that equity holders default at state B. There, the debt overhang is the entire investment benefit ε, because, if equity holders were to invest right before the short-term debt matures, debt holders would receive every dollar that new investment generates. If the cost of investment is less than $1/6\varepsilon$, for example, then there will be investment only in state G with short-term debt, while there would be investment in both states with long-term debt.

There are two lessons that we learn from this example with future investment. First, it shows that, when the firm's assets-in-place fluctuates, shorter term debt generates a more countercyclical overhang (higher overhang for low values of the firm, or weak investment incentives in future bad times). Hence, when the firm's investment opportunities are present in the future, the optimal debt maturity that minimizes overall overhang will depend on the details of future investment opportunities in different states. This idea is formally analyzed in the dynamic model in Section III.

Second, we can relate the state-dependent conditional overhang in this example back to the average date-0 overhang calculated in Section I.C. Indeed, the date-0 long-term debt overhang $2/3\varepsilon$ is just the average of the overhangs conditional on states G ($5/6\varepsilon$) and B ($1/2\varepsilon$). Similarly, the date-0 short-term debt overhang $1/2\varepsilon$ is the average of the conditional overhang in states G (0) and B (ε). Short-term debt imposes more volatile conditional overhang, but it turns out that, given the particular distribution in this example, once taking the average at date 0, short-term debt has a lower average overhang than long-term debt. The next example shows that it is possible to reverse the relative ordering of date-0 overhang by twisting the cash flow distribution, based on the idea that conditional volatilities can affect conditional overhang at different states.

E. Date-0 Investment with Conditional Volatility

Consider the date-0 investment setting as in Section I.C but modify the distribution to reduce the conditional variance of cash flows in state G. Let the new conditional probabilities given G be {1/3, 2/3, 0}; the old conditional distribution of {1/2, 1/3, 1/6} in Figure 1 is a mean-preserving spread over the new conditional distribution. Due to symmetry between states G and B in the old distribution, the new distribution features a lower variance conditional on state G than state B.

The state-contingent volatility of cash flows implies a different unconditional distribution of cash flows as of date 0. The date-0 probabilities of the final cash flows are now {1/4, 1/2, 1/4}, and using these to calculate the new face value needed to raise 8.25 with long-term debt implies that F_L is reduced to $F_L = 11$. The face value of short-term debt is unchanged at $F_S = 8.5$, a point that we explain below.

In this new example, the short-term overhang remains at $1/2\varepsilon$, because the short-term debt value 8.5 is unaffected and the firm value 8 in state B still leads to default as in the previous benchmark example in Section I.C. In fact, it is not surprising that short-term debt overhang is unchanged. To see this, note that conditional variances prevailing on date 1 govern the distribution of date-2 cash flows conditional on date-1 information, but conditional variances do not affect the payoff of short-term debt as we hold constant these future conditional market values of assets.[9] In contrast, the long-term overhang is reduced to $1/4\varepsilon$ as the outcome of zero occurs with a probability of 1/4; note that the new long-term debt face value $F_L = 11$ is below the intermediate outcome 12. Here, although short-term debt shares less risk from

[9] For instance, the increased volatility conditional on state B does not affect overhang for a small incremental investment. As long as the value of assets-in-place (after investment) is below the face value of short-term debt due before further resolution of uncertainty, all investment benefit following state B goes to short-term debt holders.

the jump down in value at date 1,[10] the short-term debt ends up taking more of the future return from date-0 investment, and hence a more severe overhang.[11]

What is the reason? As shown, the date-0 overhang is the average of future levels of overhang at different states. We observe from Section I.D that the relative severity between long- and short-term overhang is state dependent, in that long-term (short-term) debt imposes stronger (weaker) overhang in good states. Thus, the average overhang at date 0 depends on the magnitude of overhang conditional on future states.

Adjusting volatilities conditional on future states affects the relative magnitude of conditional overhang for debt with different maturities. As mentioned, because short-term debt gets repaid at date 1, the short-term overhang is not affected by date-1 conditional volatilities. However, for long-term debt, either a lower volatility in state G or a higher volatility in state B reduces overhang in the corresponding state. A lower volatility in the good state implies that there is little chance of default for a long period afterwards. In our new example, the firm with long-term debt never defaults in state G (there is zero probability of having the outcome zero), implying zero overhang. A higher volatility in state B implies that, despite a low current value, assets-in-place are more likely to increase sufficiently to repay creditors before long-term debt matures, which reduces overhang as equity can recover benefits from investment in these states. In this new example, the overhang in state B is $1/2\varepsilon$.[12] Averaging out these levels of conditional overhang, the date-0 long-term overhang is $1/4\varepsilon$.

This example appears special because the lower volatility in good times reduces the face value of long-term debt to 11 (from 12.75, without conditional volatility), which totally eliminates default when the final cash flows are 12. We present a similar example with a continuum distribution of final cash flows in the Internet Appendix to show that our result does not depend on the discrete outcomes.[13]

In sum, volatility that is higher in the bad state or lower in the good state reduces long-term debt overhang for both states at date 1, which helps average out to a lower long-term debt overhang at date 0. In contrast, the change in volatility after short-term debt matures has no effect on its overhang. This result and associated intuition are further illustrated in Section II.C in the Black-Scholes-Merton framework.

[10] The market value of short-term debt is 8.5 (8) in state G (B), while the market value of long-term debt is 11 (5.5) in state G (B). Hence, the short-term debt still shares less risk than long-term debt.

[11] One cannot construct such an example with stronger short-term overhang under the old symmetric distribution.

[12] In this example there is another indirect effect of a lower long-term debt face value under the new distribution.

[13] The Internet Appendix may be found in the online version of this article.

F. Plan for the Rest of the Paper

So far we have used simple numerical examples to illustrate our main results. The rest of the paper formalizes these results using models that are commonly used by researchers and practitioners in corporate finance. In Section II we use the Black-Scholes-Merton model to study the maturity effect of debt overhang on immediate investment, which corresponds to the results shown in Sections I.C and I.E. We then adopt the Leland framework in Section III to study the role of debt maturity on overhang when the firm with fluctuating values of assets-in-place has access to investment opportunities over time. This analysis corresponds to the setting in Section I.D where a firm chooses investment after the realization of news about its assets-in-place.

II. Immediate Investment in the Black-Scholes-Merton Model

Much of the intuition that shorter term debt enhances the incentive for investment decisions, such as Myers (1977), comes from the Black and Scholes (1973) model and the study of risky corporate debt in Merton (1974), where equity is a European call option with a strike price equal to the face value of debt to be repaid on its maturity date. This section analyzes the effect of maturity on debt overhang in a Black-Scholes-Merton setting. Although many have discussed the effect of maturity on debt overhang based on the discussion in Myers (1977), we are unaware of any existing formal analysis in the Black-Scholes-Merton setting.

A. The Black-Scholes-Merton Setting

The firm has existing assets-in-place, with current market value denoted by V_0. The asset value follows a log-normal diffusion, and its value at any future time $t > 0$ is

$$V_t = V_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma Z_t\right), \tag{2}$$

where $Z_t \sim N(0, t)$ and the volatility σ is a constant. Later we introduce state-dependent volatilities. Without loss of generality we set the risk-free rate to zero.

Following Merton (1974), the firm has a zero-coupon debt issue that matures at time t with a face value F_t; this is the firm's only debt. At time t, if the firm value V_t is below F_t, debt holders take the defaulted firm to obtain V_t; otherwise, debt holders are repaid in full by F_t. Because V_t follows a martingale, a shorter maturity of debt can equivalently be viewed as debt to be repaid after a smaller amount of resolution of uncertainty about the firm's assets. Finally, recall that we rule out both physical bankruptcy costs (i.e., the asset can be liquidated at any time for its value V_t) and renegotiation of the debt in return for a changed investment decision.

As in our numerical examples before, we consider a single investment opportunity at date 0 modeled as a small-scale expansion of existing assets.[14] We examine the effect of debt maturity on debt overhang by varying t, the time horizon to a single debt maturity. To focus on maturity only, our analysis holds constant the firm's initial leverage. More specifically, we adjust the face value F_t to hold constant the date-0 market value of debt when we vary t.

B. Stronger Short-Term Debt Overhang with Constant Volatility

We first establish a benchmark result under constant volatility: longer term debt imposes stronger overhang on the date-0 investment for a given market leverage.

It is well known in this setting that the payoff to equity holders will be reduced by debt overhang, which can be measured by the increase in the value of existing debt as a result of the scale expansion of V_0. How does debt maturity affect the amount of overhang?

Consider short-term (long-term) debt with face value F_1 (F_2) and maturity m_1 (m_2), where $m_2 > m_1$. The standard Black-Scholes calculation gives the corresponding date-0 debt value as

$$D(V_0; F_i, m_i) = V_0 (1 - N(d_i)) + F_i N\left(d_i - \sigma\sqrt{m_i}\right),$$
$$\text{where } d_i \equiv \frac{\ln\left(V_0/F_i\right) + 0.5\sigma^2 m_i}{\sigma\sqrt{m_i}}, i = 1, 2. \tag{3}$$

Debt overhang is measured by $D_V \equiv \partial D(V_0; F, m)/\partial V_0$, which captures the impact of a change in firm value on the value of existing debt. We study the wedge between two debt overhangs,

$$\Delta D_V \equiv D_V(V_0; F_1, m_1) - D_V(V_0; F_2, m_2), \tag{4}$$

where face values $F_2 > F_1$ are chosen to hold constant the initial firm leverage:

$$D(V_0; F_1, m_1) - D(V_0; F_2, m_2) = 0. \tag{5}$$

Proposition 1 formally states that ΔD_V in equation (4) is negative. Intuitively, given the same date-0 market debt value, shorter term debt always gains less from any marginal increase of the date-0 assets-in-place V_0, resulting in better equity holders' investment incentives.

[14] In the Black-Scholes-Merton setting with date-0 investment only, the firm's refinancing policy at time-t, that is, whether the firm refinances existing debt with newly issued equity or newly issued debt, is irrelevant. Recall that the debt in consideration is the only debt that the firm has, which implies that the firm will refinance this debt at its maturity date without existing claims (other than equity). Because the NPV of the date-0 investment undertaken will be known on date t and future investors break even, equity holders will recover any gain from the investment, except those going to debt holders existing at date 0.

PROPOSITION 1: *Under the Black-Scholes-Merton setting, we have $D_V(V_0; F_1, m_1) < D_V(V_0; F_2, m_2)$ whenever $D(V_0; F_1, m_1) = D(V_0; F_2, m_2)$. Thus, for a given initial debt market value, long-term debt imposes stronger overhang than short-term debt.*[15]

Because the shorter term debt is less sensitive to firm value, and there is a single immediate investment with a single maturity of debt, it takes a smaller part of the value of new investment and thus has lower overhang on investment. In the limit, if the debt matured after the investment but before any uncertainty is resolved, the value of the firm and its equity would increase by the NPV of the investment and there would be no debt overhang. This limit would be almost identical to debt that matured before the investment decision is made.

Both the example in Section I.C and the Black-Scholes-Merton setting in Proposition 1 have state-independent volatility, that is, uncertainty resolves at the same rate in good and bad states. The next subsection relaxes this assumption.

C. Stronger Short-Term Overhang with State-Dependent Volatility

Recall that in the example in Section I.E, even with a single initial investment, conditional volatilities that increase in bad states can reverse the result that shorter term debt imposes less overhang. Following this idea, we now show that a state-dependent volatility (more specifically, higher volatility given a low assets-in-place state) in the Black-Scholes-Merton setting can lead to a stronger short-term debt overhang.

Consider the following simple modification of the Black-Scholes-Merton model where short-term (long-term) debt will mature at $m_1 = 1$ ($m_2 = 2$). Suppose that the value of the firm's assets-in-place at the end of period 2 is $V_2 = V_0 \exp(\widetilde{z}_1 - 0.5\sigma_1^2 + \widetilde{z}_2 - 0.5\widetilde{\sigma_2^2})$, where $\widetilde{z}_1$ and $\widetilde{z}_2$ have zero mean and follow the normal distribution with variances σ_1^2 and $\widetilde{\sigma_2^2}$, respectively. Thus, the value of the assets-in-place on date 1 is $V_1 = V_0 \exp(\widetilde{z}_1 - 0.5\sigma_1^2)$.

To introduce state-dependent volatility, we allow the volatility $\widetilde{\sigma_2^2}$ to be dependent on date 1 assets-in-place $\widetilde{z}_1$. In particular, for some constant Q we set

$$\widetilde{\sigma}_2 = \begin{cases} \sigma_L \text{ when } \widetilde{z}_1 > Q, \\ \sigma_H \text{ when } \widetilde{z}_1 \leq Q, \end{cases} \tag{6}$$

where $\sigma_L \leq \sigma_H$. This formulation implies that asset volatility is higher in low value states (or, a negatively skewed distribution). In fact, this pattern can be generated by the existence of volatility that is not scaled with the asset value.[16]

[15] All proofs are in the Appendix.

[16] For instance, consider randomness in the fixed cost; then a fixed absolute volatility becomes a larger percentage of volatility when asset values are decreased.

It is also a natural result when the borrower's *assets* are debt contracts, for example a bank, where volatility falls in good states when debt assets become default free.

We set the long-term debt face value $F_2 = V_0 \exp\left(Q - 0.5\sigma_1^2\right)$, so that the contingent volatility is lower (higher) for regions of V_1 above (below) F_2. We have the following proposition:

PROPOSITION 2: *Adjust F_1 such that $D(V_0; F_1, 1) = D(V_0; F_2, 2)$, and suppose that $\varepsilon > 0$ is sufficiently small.*

EXAMPLE 1: *If $\sigma_L = \sigma_H = \varepsilon > 0$, that is, without contingent volatility, long-term debt imposes stronger overhang than short-term debt.*

EXAMPLE 2: *If $\sigma_H = \varepsilon > 0 = \sigma_L$, that is, with contingent volatility, short-term debt imposes stronger overhang than long-term debt.*

With two contrasting examples, Proposition 2 shows that the conditional volatility could lead to stronger short-term debt overhang for date-0 investment. In Example 2, the asset displays state-contingent $\widetilde{z}_2$ volatility, a pattern that is in sharp contrast to Example 1 with constant $\widetilde{z}_2$ volatility (which is a special case of Proposition 1). The intuition is similar to that provided in Section I.E. For short-term debt that is refinanced at date 1, whether the volatility of $\widetilde{z}_2$ is contingent or not does not affect its overhang. In contrast, for long-term debt, the volatility of $\widetilde{z}_2$ matters. To see this, after the bad realization $\widetilde{z}_1 = Q - \eta$ where η is a small positive number, the risk of $\widetilde{z}_2$ reduces overhang (as equity holders can recover some investment benefits), and this force is present in both cases with contingent and constant volatilities (the same volatility $\sigma_H = \varepsilon$). However, after the good realization $\widetilde{z}_1 = Q + \eta$, the case of contingent volatility has a lower long-term debt overhang. That is because, with contingent volatility $\sigma_L = 0$, the date-2 firm value V_2 stays constant at $V_1 = V_0 \exp\left(Q + \eta - 0.5\sigma_1^2\right)$, which is above $F_2 = V_0 \exp\left(Q - 0.5\sigma_1^2\right)$, hence there is zero overhang without future default. With constant volatility $\sigma_L = \varepsilon$, in contrast, the firm value may deteriorate at date 2, leading to potential overhang. These comparisons result in a lower long-term debt overhang in Example 2 in Proposition 2.

Proposition 2 illustrates how state-dependent volatility (higher volatility in worse states) in the Black-Scholes-Merton setting could lead to stronger shorter term debt overhang. However, the existence of state-dependent volatility is not sufficient for stronger shorter term overhang. What is general and shown in the proof of Proposition 2 is that this state-dependent volatility reduces the difference between long-term and short-term overhang. Moreover, Proposition 2 demonstrates that this effect can be sufficiently strong to overturn the positive long-short overhang wedge established in Proposition 1.

D. Debt Maturity, State-Contingent Overhang, and Conditional Volatility

In this subsection we offer another insightful way to understand the role of conditional volatility. As suggested by the numerical example in Section I.D, the relative severity of long-term and short-term overhang depends on the future state of firm value, and longer term debt imposes more (less) overhang in good (bad) states. The average of these future state-dependent overhang severities determines the date-0 investment incentives, and Proposition 1 shows that in the Black-Scholes-Merton constant volatility setting, once controlling for the date-0 market debt value, the average long-term debt overhang always exceeds the average short-term overhang.

Conditional volatility is a way to twist the state-dependent overhang to potentially deliver a greater date-0 average short-term overhang, thus reversing Proposition 1. In the Black-Scholes-Merton setting the *effective* debt maturity is inversely related to the speed of resolution of uncertainty, which is also asset volatility. From this perspective, the conditional volatility allows us to twist the *effective* debt maturities given different states. In Example 2 with contingent volatility in Proposition 2, at date-1 good states, the zero date-2 volatility implies no difference between long-term debt (which matures at date 2) and short-term debt (which matures at date 1). This minimizes the (positive) wedge in date-1 good state, that is, the excess of long-term over short-term overhang. In contrast, at date-1 bad states, short-term and long-term debts differ given the positive date-2 volatility, which preserves the negative wedge between long-term and short-term overhang. In sum, higher conditional volatilities at lower assets-in-place states can reduce the positive excess of long-term overhang over short-term overhang in good times while preserving the negative difference in bad times. For the single initial investment, this increases date-0 average overhang of short-term debt compared to long-term debt.

III. Debt Overhang with Dynamic Investment

To examine the long-horizon effects of debt maturity, we need a tractable framework with dynamic investment opportunities that goes beyond the Black-Scholes-Merton model. We have two goals for this dynamic analysis. First, we would like to study debt overhang for a firm with stochastic values of assets-in-place and with access to future investment opportunities. This will generalize the examples in Section I.D where investment is made after the realization of news about asset values. Second, all of our previous examples and Black-Scholes-Merton models have had refinancing (if any) occurring at a time when there is no other existing debt outstanding. When debt is refinanced, the incentives to refinance or default are influenced by the maturity of existing outstanding debt on that date, which is another form of debt overhang. As we show, these two crucial features (which are missing from static models) lead to interesting implications about short-term debt overhang.

A. The Setting and Valuations

Models with multiple debt issues and dynamics in the value of assets are difficult to analyze. The most tractable existing framework is based on Leland (1994b, 1998), which take as fixed parameters both the frequency of refinancing and the total amount of promised repayments of debt. Over time, as conditions change, this means that the firm keeps constant the total amount promised to debt holders at each refinancing, and does not adjust this amount to new conditions. Some effects of adjusting the amount of debt are discussed in Section III.F. In this model, equity holders always have access to funds to cover the investment costs or losses at refinancing. Default then occurs only when their incentive to inject more funding is insufficient. This allows us to eliminate issues of limited liquidity (e.g., Diamond (1991)) and focus instead on debt overhang by examining the equity holders' incentive to inject funds into the firm.

A.1. Firm Assets

Consider a firm that generates cash flows at a rate of X_t. We interpret X_t as assets-in-place, which evolve as follows:

$$\frac{dX_t}{X_t} = \tilde{i}_t dt + \sigma dZ_t. \tag{7}$$

Here, σ is the constant volatility and $\{Z_t : 0 \leq t < \infty\}$ is the standard Brownian motion. Differing from standard Leland settings, in equation (7) the growth rate $\tilde{i}_t$ is the endogenous investment decision controlled by equity holders. For simplicity, we assume that $\tilde{i} \in \{0, i\}$ takes a binary value, that is, equity holders can decide whether to invest. The investment cost is modeled as $\lambda X_t i_t dt$ because the investment benefit scales with X_t as well.

We assume a constant interest (discount) rate $r > 0$ in this infinite horizon model. If equity holders always invest, then the present value of the firm, given the current value of assets-in-place X_t , is

$$\mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)}\left(X_s - \lambda i X_s\right) ds\right] = \frac{1-\lambda i}{r-i} X_t. \tag{8}$$

Comparing this value to the value X_t/r without investment, we assume that $\lambda r < 1$, which ensures that investment at every instant maximizes the total value of the firm.

Denote the investment policy by $i(X)$, which depends on current assets-in-place X. In Proposition 3 we show that in equilibrium equity holders use a simple threshold policy, that is, invest whenever the value of assets-in-place exceeds a critical level X_i:

$$i(X) = \begin{cases} i & X \geq X_i \\ 0 & X < X_i \end{cases}. \tag{9}$$

As mentioned in Section I.A, investment can only be taken by equity holders, and future investment opportunities are lost when debt holders take over the firm from bankruptcy. This leads to an endogenous cost of financial distress. Unlike Leland's models, we impose no other exogenous costs of financial distress.

A.2. Stationary Debt Structure

The firm has one unit of debt with a constant aggregate principal face value of P. As in Leland (1994b, 1998), we assume a simple refinancing policy that governs the firm's maturity structure. Under this framework with refinancing frequency f, at each instant a constant fraction of debt, $f\,dt$, becomes due and must be refinanced to keep the amount of total debt outstanding constant. This isolates the effect of maturity from changes in the amount of debt.[17] This stationary debt structure describes a firm that smoothes out interest and principal payments to avoid spikes in refinancing activity. One immediate application of the constant refinancing rate is analysis of borrowers who for some exogenous reason have a particular debt maturity. For example, banks issue short-term deposits and have a very short debt maturity. More generally, the stationary debt structure is assumed for tractability, but is a sensible place to start. For a detailed discussion about this refinancing policy, see Section III.F.

One can show that the average debt maturity is $m \equiv 1/f$. The higher the rollover frequency f, the shorter the debt maturity. At the extreme, if f goes to infinity (so m goes to zero), then the debt represents zero-maturity demandable debt that matures immediately after the issuance.

The advantage of this setting is that, because each bond is retired exponentially, at any point in time the firm's existing bonds—including those just newly issued—are identical. Besides tractability, we adopt this framework because the overall refinancing rate is the most relevant variable to characterize a firm's debt maturity structure, and we treat this refinancing rate as a parameter. For understanding overhang, this is a reasonable treatment because the refinancing rate is essentially the frequency of repricing, and repricing to reflect the benefits of new investment is central to equity holders' incentives to invest. Thus, this framework preserves the key difference between long-term and short-term debt in regard to overhang due to wealth transfer to debt holders. Interestingly, in addition to the usual positive force of repricing to reduce overhang on investment, we will see another offsetting effect whereby shorter term debt leads equity to default earlier in bad times, which exacerbates overhang. The latter effect is closely related to rolling over debt, which we turn to next.

A.3. Rolling over Debt

The market value of the firm's debt is denoted by $D(X_t)$. In refinancing, the firm issues $(1/m)\,dt$ units of new bonds to receive total proceeds of $(D(X_t)/m)\,dt$, paying $(P/m)\,dt$ to retire maturing bonds. The market price of newly issued bonds fluctuates with assets-in-place X_t, leading to net payments to bond

[17] "Debt retirement" in this fashion is similar to a sinking fund that continuously buys back debt at par with a constant rate of repayment.

holders that we refer to as rollover gains/losses of $\frac{1}{m}\left[D(X_t) - P\right]dt$.[18] Equity holders are the residual claimants of the rollover gains or losses: any gain will be immediately paid out to equity holders and any loss will be paid off by issuing more equity at its market price. Thus, the net cash flow to equity holders is

$$X_t dt - \lambda X_t \tilde{i}_t dt + \frac{1}{m}\left[D(X_t) - P\right]dt. \tag{10}$$

The first term is the firm's cash flows, the second term is the investment cost, and the third term is the rollover loss. As emphasized by He and Xiong (2012a), when assets-in-place X_t deteriorate in value, equity holders absorb the rollover loss by issuing additional equity to prevent bankruptcy, and this loss is amplified by the rollover frequency $f = 1/m$. Equity holders are willing to inject cash to repay the maturing debt holders as long as the option value of keeping the firm alive (and hence choosing to default later) justifies the expected rollover losses. This leads to default when the equity value drops to zero, which occurs when the firm's assets-in-place X_t drops to an endogenously determined threshold X_B.

A.4. Valuations and Optimal Policies

The debt value satisfies the equation

$$rD(X) = i(X)XD'(X) + \frac{\sigma^2}{2}X^2 D''(X) + \frac{1}{m}\left(P - D(X)\right). \tag{11}$$

The left-hand side is the required return for debt, which equals the expected increment in the debt value on the right-hand side. The first two terms capture the fluctuation in X_t in equation (7). The third term is the change in debt value due to retirement: a fraction $\left(1/m\right)dt$ of debt matures, with the valuation change being the principal payment P minus the bond value before retiring.

We need boundary conditions to solve equation (11). Firms with extremely profitable assets-in-place $X = \infty$ never default and the default-free debt value is $p \equiv P/(1 + mr)$. From now on we treat the default-free debt value p as the primitive parameter (instead of the stated principal value P). On the other hand, equity defaults when $X = X_B$, and debt holders receive the firm with a value of $D(X_B) = X_B/r$ without future investment (there is no exogenous bankruptcy cost). One can formally show that $p > X_B/r$, that is, on the date of default there is a loss to debt holders.[19]

[18] Because of zero-coupon debt, discounting implies that the firm always incurs rollover losses. Rollover gains could occur if we instead assumed a bond issued at par by setting a coupon rate higher than r. Whether rollover gains are possible or not is not essential to our analysis. As shown in He and Xiong (2012a), the key is that increased rollover losses for lower values of assets-in-place increase equity holders' incentive to default.

[19] This can be seen by a standard real option argument. Suppose that $X_B \geq rp$. Then $D(X_B) = p$ and the debt is riskless. With the option to default, equity holders must incur strictly negative cash

Equity holders' value $E(X)$ satisfies the equation

$$rE(X) = \max_{i_t \in \{0,i\}} X + i_t X E'(X) + \frac{1}{2}\sigma^2 X^2 E''(X) - \lambda i_t X - \frac{1}{m}\left(P - D(X)\right). \quad (12)$$

We have omitted the optimal default policy here; equity holders default at some endogenous level X_B and receive zero. The optimization in equation (7) with respect to i_t leads to an investment policy in equation (9). The next proposition verifies the optimality of the threshold investment strategy, and gives debt and equity values as solutions to equations (11) and (12), respectively.

PROPOSITION 3: *There exists a unique X_i with $E'(X_i) = \lambda$ so that the optimal investment policy is given by equation (9). Given X_i and X_B, the debt value is*

$$D(X) = \begin{cases} p + A_1 X^{-\gamma_1} & \text{if} \quad X \geq X_i \\ p + A_2 X^{-\gamma_2} + A_3 X^{\delta_2} & \text{if} \quad X_B < X < X_i \end{cases},$$

and the equity value is

$$E(X) = \begin{cases} \frac{X(1-\lambda i)}{r-i} - p + B_1 X^{-\gamma_3} - A_1 X^{-\gamma_1} & \text{if} \quad X \geq X_i \\ \frac{X}{r} - p + B_2 X^{-\gamma_4} + B_3 X^{\delta_4} - A_2 X^{-\gamma_2} - A_3 X^{\delta_2} & \text{if} \quad X_B < X < X_i \end{cases},$$

where constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_2, \delta_4, A_1, A_2, A_3, B_1, B_2,$ and B_3 *are given in the Appendix.*

The expression for equity value is intuitive. When the firm invests $X \geq X_i$, the equity value is the firm value that would prevail if the firm always invested, $X(1-\lambda i)/(r-i)$ in equation (8), minus the default-free debt value p, with the adjustment for potential future default and stopping investment (at least temporarily). Outside the investment region $X_B < X < X_i$, the equity value is the firm value without investment X/r minus the default-free debt value p, taking into account both potential default and entering the investment region again in the future.

Finally, the endogenous investment threshold X_i satisfies $E'(X_i) = \lambda$, and the endogenous default boundary X_B satisfies the smooth-pasting condition $E'(X_B) = 0$. The detailed equations and steps in solving for these two endogenous variables are given in the Appendix.

B. Optimal Debt Maturity

Recall that the example in Section I.D illustrates the following idea: For future investment opportunities, short-term debt hurts the firm's incentives to

flows at X_B. From (11), since equity holders can set $\tilde{i} = 0$, at X_B the cash flow for equity is at least $X_B - (P - p)/m = X_B - rp > 0$, a contradiction.

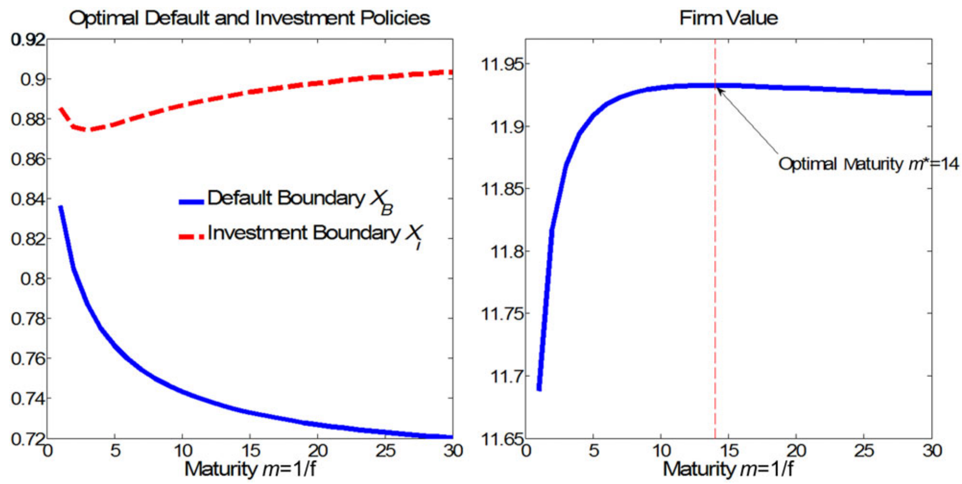

Figure 2. Optimal default policy (solid line) and investment policy (dashed line) and firm values for different debt maturities. The parameters are $r = 10\%$, $\sigma = 15\%$, $i = 7\%$, $\lambda = 9$, $X_0 = 1$, and $D_0 = 10$.

invest, especially in bad times. However, long-term debt imposes a less state-contingent overhang and reduces the firm's investment incentives, especially in good times (relative to short-term debt). As we illustrate now, this trade-off generally leads to an interior optimal maturity choice.

B.1. Trade-Off of Long-Term versus Short-Term Overhang

We choose $r = 10\%$, $\sigma = 15\%$, $i = 7\%$, and $\lambda = 9$. We normalize the date-0 assets-in-place to $X_0 = 1$, and set the target date-0 debt value of $D_0 = 10$. The left panel of Figure 2 graphs the optimal investment and default policies, and the right panel graphs the date-0 firm value; both are plotted against debt maturity m. As before, to fix date-0 debt value D_0, when varying maturity m we search for the default-free debt value p so that $D_0 = 10$ always.[20]

In this model, because shorter term debt requires equity holders to absorb greater rollover losses (incurring higher financing costs) when the firm's assets-in-place deteriorate, equity holders default earlier as they refuse to subsidize debt holders, a symptom of debt overhang. This result can also be seen by observing that, because shorter term debt does not share as much risk, it leads to more volatile equity value and hence equity holders' default option falls "into the money" more often. Graphically, in the left panel of Figure 2 we observe that the default boundary X_B rises (hence earlier default) for shorter debt maturity m. As default destroys future investment opportunities, earlier default caused by short-term overhang hurts firm value.

[20] Though not reported, in our example p is increasing in debt maturity to compensate for the greater default risk associated with longer term debt.

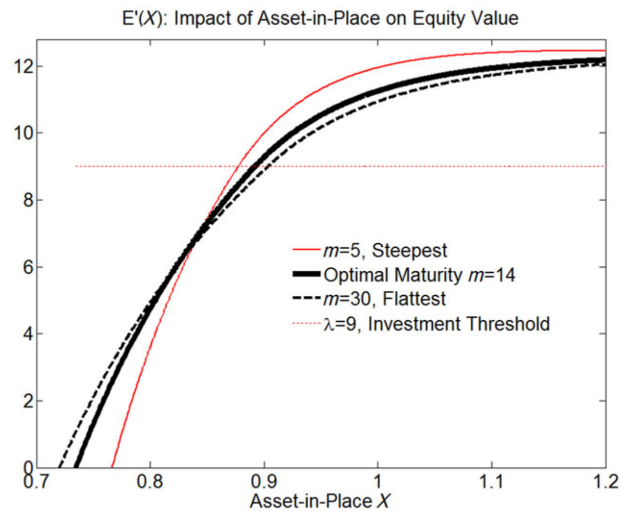

Figure 3. The impact of assets-in-place on equity value $E'(X)$ for debt maturities $m = 5$ (thin solid line), $m = 14$ (thick solid line), and $m = 30$ (thin dashed line). The investment cost $\lambda = 9$ is also plotted (flat, thin, dotted line) so that the investment threshold X_i satisfies $E'(X_i) = \lambda$. The parameters are $r = 10\%$, $\sigma = 15\%$, $i = 7\%$, $\lambda = 9$, $X_0 = 1$, and $D_0 = 10$.

Now focus on investment policy. In the left panel of Figure 2, we find that the investment threshold X_i first decreases with debt maturity for m that is below about two, then increases with debt maturity afterwards. We devote Section III.D to discuss the range of very short maturity debt (i.e., m below about two) where shorter maturity reduces investment incentives. In this section, we focus on the increasing region in which shorter debt maturity improves investment incentives.

For debt maturity m above about two, equity holders are more reluctant to invest (a higher threshold X_i) when facing debt with longer maturity (a greater m). Relative to short-term debt, although the long-term debt holders—due to less frequent repricing—share more losses with equity holders when assets-in-place deteriorate, they also share more gains given good news. Consequently, as more investment benefit goes to debt holders (increased overhang), equity holders set a higher investment threshold X_i with longer term debt.

The combination of these two forces (one on default policy, and the other on investment policy) leads to an interior optimal maturity choice ($m^* = 14$) that maximizes initial firm value, as shown in the right panel of Figure 2. To further illustrate the mechanism, Figure 3 plots the marginal impact of the firm's assets-in-place on equity value, that is, E$'$ (X) for $m = 5$ (thin solid line), $m = 14$ (thick solid line), and $m = 30$ (thin dashed line). We can directly compare the equity holders' investment incentive E$'$ (X) to the investment cost λ (flat dotted line). Firms with shorter debt maturity $m = 5$ have the steepest E' (X) curve: they invest early once E' (X) crosses λ, but also default early when E' (X) hits zero. As shown, the flatter equity holders' investment incentive E' (X)

under a longer debt maturity $m = 30$ gives the opposite effect: the firm invests late and defaults late. The curve with $m = 14$ balances these two forces and delivers the highest firm value.

Underlying the pattern of "the shorter the debt maturity, the steeper the equity holders' investment incentives" is the insensitivity of short-term debt value with respect to firm value, due to more frequent repricing. This translates to a greater volatility of debt overhang of short-term debt, a result consistent with the example in Section I.D. As shown in Figure 3, equity holders with $m = 5$ have lower $E'(X)$ for low values of assets-in-place and therefore default earlier, but also have higher $E'(X)$ for high values of assets-in-place, which fosters efficient investment. Hence, although not sharing gains makes short-term debt better at preserving equity's incentive to invest in good times, not sharing losses in bad times pushes equity holders to default, eliminating future investment opportunities. In contrast, equity holders with $m = 30$ have worse investment incentives in good times, but they are also willing to hold on longer to retain future investment opportunities, since long-term debt shares more losses in bad times.

B.2. Further Discussion

Before we move on to the next subsection, recall that in the left panel of Figure 2 both the investment and default thresholds decrease with debt maturity for very short maturity (below about two years). In this range, even shorter term debt presents a double evil—firms are not only more likely to default but also less likely to invest. This result is related to the assumption of intertemporally linked investment, a topic that we discuss in Section III.D. There, we show that, when the investment cost λ is sufficiently small, extremely short-term debt (i.e., $m \to 0$) hurts the firm's investment incentives.

Another interesting question is whether the optimal maturity in the right panel of Figure 2 can take the extreme corner values, that is, $m^* = 0$ or $m^* = \infty$. The above discussion implies that, when the investment cost λ is sufficiently small, the optimal maturity will be away from zero, that is, $m^* > 0$, simply because extremely short-term debt $m \to 0$ hurts both investment and default incentives. In fact, when the investment cost is $\lambda = 0$, we know that the optimal maturity is $m^* = \infty$. This is because, without investment cost, there is no concern of overhang in good times (the firm always invests), but in bad times bankruptcy is costly due to the loss of all future investment opportunities. As shown in Section III.E, infinite maturity $m^* = \infty$ always helps by postponing inefficient default. We leave for future research the determination of whether $m^* = 0$ is optimal given a sufficiently high investment cost parameter λ.

C. State-Dependent Investment Opportunities

The discussion of the higher volatility of investment with shorter term debt suggests a role of state-dependent investment opportunities. Because short-term debt is better at preserving equity holders' investment incentives in good times, a firm with sufficiently good investment opportunities that appear primarily in good times should use shorter term debt for better investment incentives. This section further explores this idea.

C.1. Fixed Investment Cost and Optimal Policies

Keep the same binary investment technology (either invest or not) as in Section III.A, but modify the investment cost as a combination of fixed cost part ρ and variable cost part Δ:

$$\lambda\left(X_t;\rho,\Delta\right)=\rho+\Delta\lambda i X_t. \tag{13}$$

We can broadly interpret the case in which $\rho<0$ as a fixed investment benefit, because the firm receives $-\rho$ as a flow benefit from investing. When $\rho=0$ and $\Delta=1$, we are back to our base model above with a pure variable cost. To isolate state dependence of investment opportunities from overall profitability, we posit a cost structure so that on average the firm value remains unchanged; for instance, $\rho>0$ and $\Delta<1$.

The structure in equation (13) allows us to use the fixed investment cost part $\rho>0$ to proxy for the state-dependence of investment opportunities. To show this, we demonstrate that a higher ρ implies a higher correlation between investment opportunities and assets-in-place. Recall that we always keep the investment benefit as purely variable, that is, X_t grows at rate i by investing. Therefore, the NPV of investing always from now on is (using the similar calculation as in equation (8))

$$\underbrace{\left(\frac{X}{r-i}-\frac{X}{r}\right)}_{\text{Investment benefit}}-\underbrace{\left(\frac{\Delta\lambda i X}{r-i}+\frac{\rho}{r}\right)}_{\text{Investment cost}}=\frac{i\left(1-\Delta\lambda r\right)}{r\left(r-i\right)}X-\frac{\rho}{r}. \tag{14}$$

On the left-hand side, the first parentheses give the increment of the firm's present value by always investing, and the second parentheses give the present value of the total investment cost. The difference is the NPV of investment (always), given on the right-hand side.

Equation (14) suggests that, relative to the base case of $\rho=0$ and $\Delta=1$, the case of $\rho>0$ and $\Delta<1$ makes the investment NPV more sensitive to X_t, that is, increases the correlation between investment opportunities and assets-in-place. Similar but exactly opposite reasoning implies that the firm with $\rho>0$ and $\Delta<1$ has worse investment opportunities for lower assets-in-place.[21] Hence, from now on we refer to ρ as the correlation between the firm's assets-in-place and investment opportunities. The higher the ρ, the better the investment opportunities in good times. In the Appendix we give the valuations and optimal policies for this extension.

[21] With only a fixed cost, even an all-equity firm stops investing when the variable investment benefit goes below the fixed cost for sufficiently low assets-in-place (e.g., for X_t close to zero).

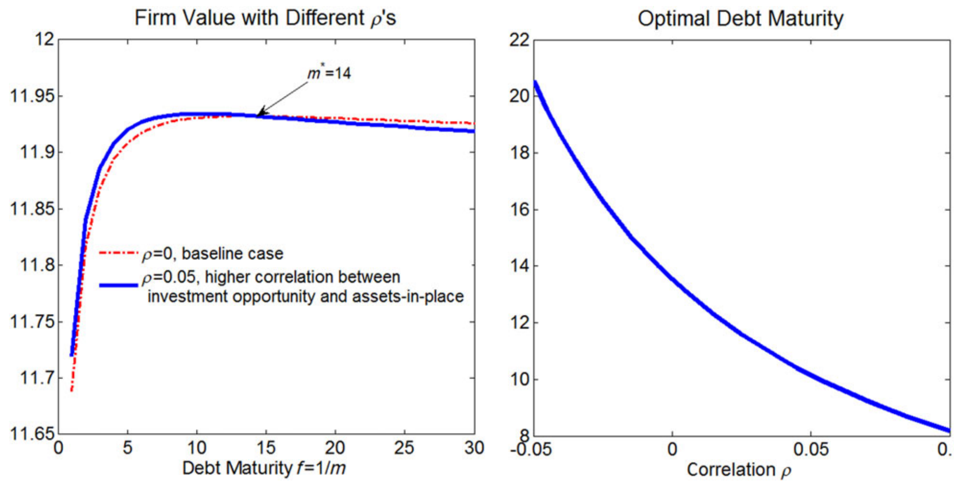

Figure 4. Optimal debt maturity as a function of state dependence of investment opportunities. The left panel plots the firm value against debt maturity, for both the baseline case ($\rho = 0$, dashed line) and the case with a higher correlation between investment opportunities and assets-in-place ($\rho = 0.05$, solid line). The right panel plots the optimal debt maturity as a function of ρ. The parameters are $r = 10\%$, $\sigma = 15\%$, $i = 7\%$, $\lambda = 9$, $X_0 = 1$, and $D_0 = 10$.

C.2. Optimal Debt Maturity and State-Dependent Investment Opportunities

The insight developed in Section III.B suggests that the firm should use shorter term debt if its investment opportunities are better for higher values of assets-in-place, that is, higher ρ. The logic is simple. In Figure 3 we show that shorter term debt leads to lower overhang for higher values of assets-in-place. Therefore, shorter term debt is particularly good at motivating equity holders to invest in good times, if indeed the firm has good investment opportunities around that time. In bad times, there will be fewer investment opportunities and the high overhang from shorter term debt will imply little loss from passing up profitable investments.

To illustrate this point, we start with the base model with a pure variable cost (i.e., $\rho = 0$ and $\Delta = 1$), and find the optimal maturity m^* that achieves the highest firm value V^*. We then consider $\rho > 0$, and control for total firm value by adjusting downward the variable cost part Δ appropriately. Call the resulting variable cost Δ_ρ. As shown in the left panel of Figure 4, this requires that the new firm value (as a function of debt maturity m, the solid line) under the cost structure (Δ_ρ, ρ) crosses the baseline firm value curve (the dashed line) at m^*.

We are interested in the optimal maturity that maximizes firm value under (Δ_ρ, ρ) with $\rho > 0$. Because firm value has a negative slope at the original optimal maturity m^* in the left panel of Figure 4, the optimal maturity with a higher $\rho > 0$ must be shorter than m^* (with $\rho = 0$).

Denote the resulting optimal maturity, as a function of ρ, by $m^*(\rho)$. The procedure described above allows us to graph the optimal maturity $m^*(\rho)$ in the right panel of Figure 4. As expected, when the firm has greater investment opportunities for high assets-in-place, the longer term debt overhang hurts firm value more severely, and as a result the firm should adjust its optimal debt maturity toward shorter term debt.

This result has important empirical implications. The debt of a growth firm should have a shorter term, only if we define a growth firm as one with substantial uncertainty about its investment opportunities *and the value of its new investment projects is highly correlated with the value of its existing investment projects*. In comparison, if future opportunities are known and current asset returns are not very informative about these opportunities, then, for a given amount of leverage, firm value is higher with longer term debt. Firms with investment opportunities that are most important in bad times, such as mature firms for which maintenance investment is needed to replace unexpectedly high depreciation and there is little learning about other aspects of profitability, maximize value with even longer term debt. This is very different from the existing view that firms with substantial future investment opportunities should choose shorter term debt.

D. Short-Term Debt Overhang and Intertemporally Linked Investment

At the end of Section III.B we point out that, in the left panel of Figure 2, short-term debt may be doubly evil: for maturity below two, firms with shorter term debt may default earlier and be less likely to invest.

To understand this result, we stress an important property of our investment technology. In our dynamic setting laid out in Section III.A.1, equation (7) implies that investment benefits are intertemporally linked. Consider the hypothetical situation that, starting from next period, the firm may lose its future investment opportunities with an exogenous probability $\pi \in [0, 1]$, but keeps investing in the future with probability $1 - \pi$. One can interpret π as the firm's *exogenous* default probability.

Investment today improves the current value of assets-in-place X_t to $X_t(1 + idt)$ at the end of today; but what about the market value of assets-in-place? Because the value is $\frac{1-\lambda i}{r-i} X_t(1 + idt)$ with future investment but only $X_t(1 + idt)/r$ otherwise (recall equation (3)), the firm value at the end of today, given the exogenous default probability π, is

$$\left[\pi \frac{1}{r} + (1 - \pi)\frac{1 - \lambda i}{r - i}\right] X_t(1 + idt). \tag{15}$$

Importantly, this value is decreasing in the default probability π. In our setting with log-normal structure, investment decisions at all periods enter total firm value in a multiplicative fashion, and hence a high future investment policy boosts equity holders' investment incentive today. The positively connected investment incentives across periods are natural, and particularly relevant in situations with staged investments.

In our model, the probability of future default π is endogenous, and moreover, increasing in equity holders' default threshold X_B. In the left panel of Figure 2, the ultrashort debt maturity leads equity holders to set a high default threshold X_B. Hence, current investment that will not grow in the future becomes less profitable. Put differently, short-term debt and associated default eliminates future firm growth, which reduces the total payoff of current investment.

In Section III.B we see another opposite force, through which short-term debt improves equity holders' investment incentives due to more frequent repricing. Which force prevails depends on parameters. In Proposition 4 we consider the limiting case in which debt approaches demandable debt with zero maturity, and we provide a sufficient condition under which the effect of eliminating future growth dominates.

As we point out in the Appendix, this *sufficient* condition always holds for low but positive investment cost, that is, when λ is sufficiently small. To see this, recall that $E'(X_i) = \lambda$ while $E'(X_B) = 0$; hence λ also captures the distance between investment and default. If this distance is short, then the effect of eliminating future growth dominates. By interpreting investments that should be made even when close to default as maintenance, this result essentially says that the shorter the debt maturity, the sooner the firm defaults, and the earlier the firm stops maintaining its assets.

PROPOSITION 4: *Fix the date-0 market value of debt, and let debt maturity approach zero, that is, $m \to 0$. We have the following results:*

(1) *The default threshold X_B converges to rp.*
(2) *When the primitive parameters satisfy the sufficient condition provided in the Appendix, the investment threshold X_i is higher (making the firm less likely to invest) for shorter debt maturity.*

Proposition 4 establishes that debt overhang can exist even if firms issue a sequence of demandable debt contracts. Moreover, note that X_B/r is the limit of firm value recovered at default by debt holders. The first result in Proposition 4 therefore says that, for ultra-short-term debt, what debt holders recover in fact converges to bonds' default-free value p.[22] Therefore, these debt securities are almost riskless, but still impose overhang!

This result is interesting because, from the standard static model with a single investment opportunity, the logic of Myers (1977) implies that riskless debt leads to no overhang. However, in our dynamic model with intertemporally dependent investment incentives, these *almost* risk-free debt securities impose strong overhang on equity holders. What drives this result?

The key mechanism is truncation of future investment opportunities in (future) bad states where the borrower chooses to default. Although lender losses

[22] A similar result for ultra-short-debt is obtained by Leland and Toft (1996) in a model without investment opportunities but with bankruptcy costs.

are very small, this truncation hurts equity holders' investment incentives today, given intertemporally linked investment opportunities. In a static model, truncation is just payment default and necessarily implies risky debt. In a dynamic investment model, the significance of truncation of future investment is not related to whether debt holder losses are large. In fact, if anything, safer debt securities require more truncation, because to maintain safety debt holders should take over the firm at a higher value of assets-in-place. This is closely related to an extreme situation considered in Leland (1994a), where the firm issues "protected debt" contracts whose holders can take over the firm whenever the firm's value drops to the debt default-free value. By definition the debt is risk-free; however, one can show that equity holders' investment incentive right before default is $E'(X) = 1/r$ (setting $\pi = 1$ in equation (15)),[23] which is below the first-best investment incentive $\frac{1-\lambda i}{r-i}$. As a result, protected riskless debt imposes overhang on intertemporally linked investment.

E. What if the Investment Opportunity Is Only in the Future?

We have seen one drawback of short-term debt, namely, inefficiently early default, which eliminates future investment opportunities. To study this effect in isolation, we consider the simplified model without interim investment, that is, $i = 0$. Instead, the firm is waiting for a second stage for expansion, after which there is no future uncertainty. This expansion stage arrives as a Poisson event with intensity $\xi > 0$ for equity holders to improve assets-in-place X_t to θX_t at no cost, with $\theta > 1$. We set the expansion cost to zero so that equity holders are always willing to expand. This eliminates debt overhang in the expansion stage, and allows us to focus only on the debt overhang effect of default before the expansion stage.

Because default at X_B eliminates the future investment opportunity, our modeling is isomorphic to a standard (exogenous) bankruptcy cost as in Leland (1994a). Hence, maximizing firm value is equivalent to inducing default as late as possible, that is, minimizing the default boundary X_B. The derivation of debt value, equity value, and endogenous default boundary is standard (see the Appendix). The endogenous default boundary can be derived in closed form as:

$$X_B = \frac{rp\gamma_5(m)}{\gamma_5(m) - \gamma_6 + (1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}, \tag{16}$$

where $\gamma_5(m)$ and γ_6 are given in equation (A3) in the Appendix. We further show that all else equal, X_B is increasing in p, as equity holders default earlier for a greater debt burden, and X_B is decreasing in maturity m, so that shorter term debt leads to earlier default. The next proposition gives the stronger result that X_B is decreasing in maturity m, even if accounting for the firm adjusting the promised debt value p to control for leverage D_0.

[23] At the default boundory $\underline{X}_B$ we always have $E'(X_B) = 0$, but one can show that $E'(X) = 1/r$ for $X > X_B$ when $m = 0$.

PROPOSITION 5: *Fix the date-0 debt value. Equity holders always choose p so that the date-0 debt value is increasing in p (i.e., $\partial D_0 / \partial p \geq 0$), and the default boundary X_B is decreasing in m. Hence, extremely long-term debt (i.e., $m = \infty$) is preferable.*

Proposition 5 notes that, at the optimum, the market value of debt always increases in the promised payment p . This excludes what might otherwise happen due to a debt "Laffer curve," where raising an already excessive promised payment reduces the debt value by inducing excessively earlier default. Equity holders who maximize date-0 firm value avoid this wrong side of debt "Laffer curve,"[24] and with the aid of this observation we are able to prove that $dX_B / dm < 0$, that is, equity holders default later with longer term debt.

This result formalizes the intuition that "long-term debt is better at preserving the firm's incentives to stay alive for future investment." Also, one can show that, in a similar spirit to Section II with immediate investment, if the investment opportunity is only present at date 0, then shorter term debt is preferable in reducing overhang (the proof is available upon request). These two extreme cases (one only with future investment, the other only with immediate investment) are the forces behind the trade-off shown in Section III.B for firms with interim investment opportunities.

F. Discussion of a Changing Refinancing Policy

Refinancing policy, which affects future overhang when issuing new debt, is one of the natural ingredients of an optimal debt maturity structure. Following Leland (1994b, 1998), for tractability we perform our analysis based on a stationary debt structure with a constant refinancing rate $f = 1/m$. Because the option to default and repricing are the reasons that maturity influences debt overhang, this dynamic framework with stochastic values of assets-in-place captures important aspects of how debt maturity influences equity holders' incentives to invest over time.[25] Moreover, our analysis applies to the situation in which the firm's debt maturity structure is largely determined by other considerations (e.g., commercial banks with demand deposits).

What if firms have flexibility in adjusting their refinancing policies, instead of a constant refinancing policy? Determining a desirable policy with a variable debt maturity requires knowledge of the effects of a fixed maturity. A full analysis of the case of flexible refinancing policies is intractable using existing

[24] That is, there exists some default-free debt value $\bar{p}$ that produces a maximum market value of the date-0 debt. Equity holders would never promise debt holders more than $\bar{p}$ because they could raise the same amount by promising them less and defaulting less (both of which increase the payoff of equity holders).

[25] The convenient feature of this setting is that, looking forward, the model is always identical, because the firm keeps the total amount of promised payments constant over time. What varies is the amount raised from this promise (and this leads to a time-varying shortfall of funds that is larger when default risk is higher). In practice, this is primarily about units of measurement of default premia. Whether the default premia are represented by a larger shortfall or higher future promises does not drive our qualitative results.

models and beyond the scope of this paper.[26] Nevertheless, we provide some discussion that may be helpful in understanding the robustness of our results to other refinancing policies.

F.1. Robustness of State-Contingent Short-Term Debt Overhang

Our key result that short-term debt imposes more (less) overhang in bad (good) times should be qualitatively robust. Take our model, but suppose that the firm can freely set the refinancing rate f for newly issued bonds as the assets-in-place value X_t fluctuates. If this were possible, the firm could, as part of an explicit contract, adjust its debt maturity structure dynamically. For better investment/default incentives, we expect the refinancing rate f for newly issued bonds to be increasing in assets-in-place X_t,[27] so that the firm is targeting a move toward a shorter term (longer term) maturity structure when the value of assets-in-place is high (low) on average. Having said that, because the firm's assets-in-place fluctuates unexpectedly, and it takes time to fully adjust the firm's overall debt maturity, having short-term (long-term) debt only in good (bad) times is not achievable. Moreover, the unexpected fluctuation of X also implies that it is not optimal to take some extreme refinancing policies, say $f(X) \to 0$ for sufficiently low X. For this reason, the qualitative result that short-term debt imposes more (less) overhang in bad (good) times should remain in the more general model.

The above discussion suggests that debt with state-contingent maturities, especially bonds with automatically reset longer maturity in bad times and shorter maturity in good times, is value-improving. In practice, we do observe this favorable state-contingency in the call feature of some long-term debt.[28] Hence, firms with state-independent future investment opportunities can improve investment incentives by using callable debt.

F.2. Dilution Issue when Lengthening Maturity in Bad Times

Another important issue relates to dilution, such as transfers to other debt holders if equity holders were to choose to lengthen the maturity of new debt

[26] Existing studies of dynamic adjustments to leverage by firms with investment decisions assume that debt maturity is fixed. Moyan (2007) assumes that long-term debt can never adjust its future payments while short-term debt can and will always adjust to its optimal leverage. Titman and Tsyplakov (2007) follow Fischer, Heinkel, and Zechner (1989) to assume that a firm has to call back its entire outstanding debt whenever adjusting its capital structure. This (fairly strong) assumption is needed for tractability, by eliminating the chance to dilute existing debt, but may not be close to reality. The next section describes the importance of potential dilution.

[27] The refinancing rate of newly issued bonds may also depend on the maturity structure of the firm's exiting debt.

[28] Bodie and Taggart (1978) suggest that callable long-term debt can alleviate overhang. However, the authors still find it puzzling that firms do not simply roll over short-term debt, which suggests that they do not realize that rolling over short-term debt may worsen overhang in some states. An earlier draft of our paper examined callable bonds and cash holdings (in a setting where managerial incentives were impaired by large cash holdings).

during bad times. Fixing the amount promised in the future, lengthening maturity today with existing debt in place increases today's rollover losses to equity—simply because longer term debt has a lower price than shorter term debt with the same face value. This also implies that moving to longer maturity subsidizes existing soon-to-mature debt (by equity holders), as the policy gives existing debt holders effective seniority (i.e., the timing of receiving money back) relative to new incoming debt holders with longer maturity.[29] The greater rollover losses in bad times in turn increase equity holders' default incentives for firms close to default, suggesting that dynamic policies of lengthening debt maturity and increasing rollover losses during bad times may be neither desirable nor in the ex post interest of equity. In fact, for similar reasons, Titman and Tsyplakov (2007) find that equity holders may lack incentives to adjust firm leverage downward in bad times, although it is optimal to do so from the firm's perspective. See Diamond (1991), Diamond and Rajan (2001b), and Brunnermeier and Oehmke (2013) for models where debt becomes shorter term in bad times with a higher probability of default, which would further increase its overhang.

This argument that equity is averse to increasing debt maturity (at market prices) in bad times also lends support to the generality of the result that the policy of (a sequence of) ultrashort riskless debt imposes overhang on intertemporally linked investment opportunities (see Section III.D). However, without formal analysis, it is hard to be more precise about what would happen once we allow for flexible refinancing policies. We await future research on this topic.

F.3. Flexibility of Short-Term Debt

Myers (1977) proposes that short-term debt could provide "*for continuous and gradual renegotiation, in which the firm can in principal shift at any time back to all-equity financing.*" This suggests that short-term debt offers flexibility for the firm to adjust its capital structure to avoid overhang. The benefit would be the shorter time for all or most debt to mature before either switching to all equity or negotiating with lenders for the benefits that equity holders obtain from injecting more cash. A related result is presented in a model by Brunnermeier and Yogo (2009), where the risk-free short-term debt allows the firm, at maturity, to switch to long-term debt financing, which reduces the risk of a costly bankruptcy. This result holds because long-term debt defers bankruptcy and very long-term debt is not allowed in their model. Thus, continuing to issue short-term debt until the long-term debt is needed allows a longer effective maturity than the longest possible maturity available to the firm initially.

As examples of dynamic adjustment of maturity, both ideas rely on the assumption that there exists a time such that some news about investment

[29] This dilution is analyzed in Brunnermeier and Oehmke (2013) in a model with no investment after debt is issued, implying that long-term debt is efficient. Under certain conditions, equity holders prefer to issue a unit of new short-term debt, which should be priced at a higher level than a unit of otherwise equal long-term debt. In essence, this new short-term debt dilutes the value of existing long-term debt holders.

opportunities or future default risk arrives before existing debt suffers any risk of default. Otherwise, as analyzed in our model with interim investment opportunities, short-term debt that is about to mature soon will impose overhang, which reduces equity holders' incentives to invest or change to longer maturity—to the extreme, equity holders may even default. In addition, if the merit of short-term debt is to allow firms to get back to all-equity financing as quickly as possible, then the primitive reason for the firm to raise debt in the first place becomes important. Indeed, through calibration, Moyan (2007) finds that, in a setting where firms raise short-term leverage to shelter taxes from improved profits before investment decisions, short-term debt leads to overhang as strong as that of long-term debt despite the fact that the firm can adjust short-term debt every period. What our analysis adds is that in an uncertain economy it can be very risky to rely on short-term debt to provide increased flexibility in future refinancing decisions. Related to this point, Dangl and Zechner (2008) show that short-term debt may give equity incentives to reduce leverage when the firm performance is poor, but not when the firm is close to bankruptcy.

IV. Conclusion

Debt maturity influences investment incentives in a more nuanced way than suggested by existing analysis. By definition, investment incentives are weak (and debt overhang is severe) when very little of the return from investment accrues to equity. For a single immediate investment, we show in a Black-Scholes-Merton model that shorter term debt is less sensitive to increased firm value from a new investment. This provides intuition for why shorter term debt may impose less overhang—the difference between the total return from investment and the part accruing to equity is the change in the value of debt. When investment opportunities are present in the future, this intuition is incomplete. Less risk shared with existing shorter term debt makes equity value and debt overhang more volatile, which affects future investment incentives.

We illustrate three ways in which shorter term debt can impose stronger overhang. First, when the volatility of firm value is sufficiently higher in bad times than good times, shorter term debt can lead to higher overhang even for a single immediate investment decision taken just after the debt is issued. Second, in a dynamic setting with future investment opportunities, the reduction in equity value due to the combination of bad times and shorter term debt is so large that equity holders' investment incentives suffer greatly, in which case equity holders may choose to default earlier. Third, because shorter term debt induces earlier future default and elimination of future growth, it hurts equity holders' incentives to invest (maintain) today when investment benefits are intertemporally linked.

An interesting application combines all of our results. For reasons other than the effects on debt overhang, banks and other financial institutions issue short-term debt such as deposits, which matches well with our exogenous constant refinancing structure, and fund debt contracts such as loans, which implies high asset volatility in bad times. Our model suggests that the effects of debt overhang in bad times will be extremely large for banks. This is for reasons other than the risk of runs and asset illiquidity leading to severe short-term debt overhang as in Diamond and Rajan (2001b). Adding our results to theirs suggests that the debt overhang problem for banks may be very severe.

The link between investment incentives and debt maturity is important for firms where future investments are important. Besides offering several testable implications for future empirical research, our paper suggests that managers who understand only one part of the effect of debt maturity on investment incentives could make poor choices of debt maturity structure.

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Appendix A

Proof of Proposition 1: For simplicity denote V_0 by V, and without loss of generality set $\sigma = 1$ (which amounts to an absolute time change). Given m and F, and using $n()$ to denote the density function for normal distribution, it is easy to calculate

$$D_F(V, F, m) = \frac{n(d)}{F\sqrt{m}} > 0,\ D_m(V, F, m) = -\frac{V n(d)}{\sqrt{m}} < 0, \tag{A1}$$

and

$$D_{VV}(V, F, m) = -\frac{n(d)}{V\sqrt{m}} = -\frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(\ln\left(V/F\right)+0.5m\right)^2}{2m}\right)}{V\sqrt{m}} < 0. \tag{A2}$$

For short-term debt with maturity m_1 and long-term debt with maturity m_2, where $m_2 > m_1$, equation (A1) implies that, to maintain the date-0 debt value, we must have $F_2 > F_1$. Define the difference between these two debt values as $\Delta D(V) \equiv D(V, F_1, m_1) - D(V, F_2, m_2)$, which, due to (A2), satisfies

$$\Delta D_{VV}(V) = -\frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(\ln\left(V/F_1\right)+0.5m_1\right)^2}{2m_1}\right)}{V\sqrt{m_1}} + \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(\ln\left(V/F_2\right)+0.5m_2\right)^2}{2m_2}\right)}{V\sqrt{m_2}}. \tag{A3}$$

It is easy to verify that

$$\Delta D(\infty) = F_1 - F_2 < 0,\ \lim_{x\to 0}\Delta D(x) = 0,\ \lim_{x\to 0}\Delta D_v(x) = 0.$$

Given these results, we now show that, if $\Delta D(V) = 0$ admits positive solutions, then $\Delta D(V)$ only crosses zero at most once from above; note that this result implies that $\Delta D_V(V) < 0$ whenever $\Delta D(V) = 0$, which is the desired result. Suppose that $\Delta D(V) = 0$ admits some solution. Because $\Delta D(\infty) < 0$, we can find the largest solution $\hat{V} > 0$ so that $\Delta D(\hat{V})$ crosses zero from above. Since $\Delta D_V(\hat{V}) < 0$ while $\Delta D_V(\infty) = 0$, there must exist a region where $\Delta D(V)$ is convex for $V > \hat{V}$, so that ΔD_{VV} is positive. The following lemma is useful.

LEMMA A1: *Aside from $V = 0$, ΔD_{VV}, as a function of V, can be zero at most twice.*

Proof: For $\Delta D_{VV} = 0$ we need

$$\frac{\exp\left(-\frac{\left(\ln\left(V/F_1\right)+0.5m_1\right)^2}{2m_1}\right)}{\sqrt{m_1}} = \frac{\exp\left(-\frac{\left(\ln\left(V/F_2\right)+0.5m_2\right)^2}{2m_2}\right)}{\sqrt{m_2}}.$$

Rearranging, we reach a quadratic equation for $\ln V$. The result follows easily. Q.E.D.

Given this lemma, we now have three cases to consider. In all cases we rule out the possibility of multiple solutions for $\Delta D(V) = 0$.

(1) Suppose that $\Delta D_{VV} = 0$ has no root, which implies that $\Delta D_{VV} > 0$. This cannot be true because it is inconsistent with $\Delta D_V(0) = 0$, but $\Delta D_V\left(\hat{V}\right) < 0$.

(2) Suppose that there exists only one root for $\Delta D_{VV} = 0$, which implies that $\Delta D_{VV}(V)$ is initially negative and then turns positive. Therefore, $\Delta D(V)$ is always concave before it turns convex. This implies that there will not exist another solution $V' < \hat{V}$ so that $\Delta D\left(V'\right) = 0$. To see this, note that $\Delta D(0) = 0$; then, before the earlier solution V' at which $\Delta D(V)$ crosses zero from below, we must have $\Delta D < 0$ (the bottom point somewhere between zero and V') so that ΔD is convex. This convex part takes place before the concave part between V' and $\hat{V}$, contradiction.

(3) Suppose we have two roots for $\Delta D_{VV} = 0$. Then it must be first positive, then negative, then positive; that is, $\Delta D(V)$ is first convex, then concave, and finally convex. In words, $\Delta D(V)$ can only be concave in one interval. Now, since it is easy to show that $\Delta D_V(0) = 0$, the initial convexity implies that $\Delta D(V)$ is positive for $V = 0+$. Then, for $\Delta D(V)$ to have two solutions after zero, we must have that $\Delta D(V)$ becomes concave, convex, and then concave again. This contradicts the restriction that $\Delta D(V)$ is concave only in one interval. Q.E.D.

Proof of Proposition 2: The first result follows from Proposition 1. For the second case, without loss of generality, set $Q = 0$. Consider $V_2 = V_0 \exp(\widetilde{z}_1 - 0.5 + \widetilde{z}_2 - 0.5\widetilde{\sigma_2^2})$, so that

$$\widetilde{\sigma}_2 = \begin{cases} 0 & \text{when} \quad \widetilde{z}_1 > 0 \\ \sigma = \varepsilon & \text{when} \quad \widetilde{z}_1 \leq 0 \end{cases},$$

where ε is sufficiently small. When $\sigma = 0$, the second period adds no risk, and long-term debt is identical to short-term debt with the same value and overhang. We set $F_2 = \exp(-0.5)$, so that potential second-period noise occurs exactly when the long-term debt is at the money.

We consider a perturbation from $\sigma = 0$ to $\sigma = \varepsilon > 0$, and compare the change of overhang effects on each debt. By adjusting F_1 to ensure that both debts have the same value, we aim to show that the following term (where we denote debt overhang by $OH_t = D_t'$) is positive:

$$\frac{dOH_1}{d\sigma} - \frac{dOH_2}{d\sigma} = \frac{dOH_1}{dF_1}\frac{dF_1}{d\sigma} - \frac{dOH_2}{d\sigma} = \frac{dOH_1}{dF_1}\frac{dD_2/d\sigma}{dD_1/dF_1} - \frac{dOH_2}{d\sigma}.$$

We show that raising σ from zero has no first-order effect on long-term debt value D_2, that is, $\frac{dD_2}{d\sigma}|_{\sigma=0} = 0$ while $\frac{dOH_2}{d\sigma}|_{\sigma=0} < 0$. Therefore, since $dD_1/dF_1 > 0$, we obtain our result. To show these results, we have (note that $Q = \ln F_2 + 0.5$)

$$D_2(\sigma) = F_2\int_0^\infty n(x)\,dx + \int_{-\infty}^0 \left\{\int_{-\infty}^{-x+\frac{\sigma^2}{2}} \exp\left(x - \frac{1+\sigma^2}{2} + y\right) n\left(\frac{y}{\sigma}\right)\frac{1}{\sigma}dy\right.$$
$$\left. + F_2\int_{0-x+\frac{\sigma^2}{2}}^\infty n\left(\frac{y}{\sigma}\right)\frac{1}{\sigma}dy\right\} n(x)\,dx$$
$$= F_2\int_0^\infty n(x)\,dx + F_2\int_{-\infty}^0 \left\{\exp(x)\int_{-\infty}^{-x+\frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(y-\sigma^2\right)^2}{2\sigma^2}\right)dy\right.$$
$$\left. + \int_{-x+\frac{\sigma^2}{2}}^\infty n\left(\frac{y}{\sigma}\right)\frac{1}{\sigma}dy\right\} n(x)\,dx$$
$$\overset{\textit{change variable}}{=} F_2\int_0^\infty n(x)\,dx + F_2\int_{-\infty}^0 \left\{\exp(x)\int_{-\infty}^{-\frac{x}{\sigma}-\frac{\sigma}{2}} n(t)\,dt\right.$$
$$\left. + \int_{-\frac{x}{\sigma}+\frac{\sigma}{2}}^\infty n(t)\,dt\right\} n(x)\,dx.$$

Therefore,

$$\frac{dD_2(\sigma)}{d\sigma} = F_2\int_{-\infty}^0 \left\{\exp(x)\,n\left(-\frac{x}{\sigma}-\frac{\sigma}{2}\right)\left(\frac{x}{\sigma^2}-\frac{1}{2}\right) - n\left(-\frac{x}{\sigma}+\frac{\sigma}{2}\right)\left(\frac{x}{\sigma^2}+\frac{1}{2}\right)\right\} n(x)\,dx$$
$$= F_2\int_{-\infty}^0 \left\{n\left(-\frac{x}{\sigma}+\frac{\sigma}{2}\right)\left(\frac{x}{\sigma^2}-\frac{1}{2}\right) - n\left(-\frac{x}{\sigma}+\frac{\sigma}{2}\right)\left(\frac{x}{\sigma^2}+\frac{1}{2}\right)\right\} n(x)\,dx$$

$$= -F_2 \int_{-\infty}^{0} n\left(-\frac{x}{\sigma}+\frac{\sigma}{2}\right) n(x)\,dx = -\sigma F_2 \int_{-\infty}^{0} n\left(-u+\frac{\sigma}{2}\right) n(\sigma u)\,du,$$

which is zero when $\sigma = 0$. However, since

$$OH_2(\sigma) = 1 - \int_{0}^{\infty} n(x)\,dx - \int_{-\infty}^{0}\int_{-x+\frac{\sigma^2}{2}}^{\infty} n\left(\frac{y}{\sigma}\right)\frac{1}{\sigma}dy\,n(x)\,dx$$
$$= \int_{-\infty}^{0} n(x)\,dx - \int_{-\infty}^{0}\int_{-\frac{x}{\sigma}+\frac{\sigma}{2}}^{\infty} n(t)\,dt\,n(x)\,dx,$$

and the first-order effect on overhang by raising σ is

$$\frac{dOH_2(\sigma)}{d\sigma} = \int_{-\infty}^{0} n\left(-\frac{x}{\sigma}+\frac{\sigma}{2}\right)\left(\frac{x}{\sigma^2}+\frac{1}{2}\right) n(x)\,dx = \int_{-\infty}^{0} n\left(-\frac{x}{\sigma}+\frac{\sigma}{2}\right)\left(\frac{x}{\sigma}+\frac{\sigma}{2}\right) n(x)\,\frac{1}{\sigma}dx$$
$$\overset{\text{let}\, t = \frac{x}{\sigma}+\frac{\sigma}{2}}{=} \int_{-\infty}^{\frac{\sigma}{2}} n(-t+\sigma)\,t\,n\left(\sigma t - \frac{\sigma^2}{2}\right) dt \overset{\text{let}\,\sigma=0}{=} -\frac{1}{2\pi} < 0,$$

we have proved our second result. Q.E.D.

Proof of Proposition 3: Define the following constants:

$$\begin{aligned}
\gamma_1 &= \frac{i-0.5\sigma^2+\sqrt{\left(i-0.5\sigma^2\right)^2+2\sigma^2(r+f)}}{\sigma^2} > 0,\\
\gamma_3 &= \frac{i-0.5\sigma^2+\sqrt{\left(i-0.5\sigma^2\right)^2+2r\sigma^2}}{\sigma^2} > 0,\\
\gamma_2 &= \frac{-0.5\sigma^2+\sqrt{0.25\sigma^4+2\sigma^2(r+f)}}{\sigma^2} > 0,\ \delta_2 = \frac{0.5\sigma^2+\sqrt{0.25\sigma^4+2\sigma^2(r+f)}}{\sigma^2} > 1,\\
\gamma_4 &= \frac{-0.5\sigma^2+\sqrt{0.25\sigma^4+2r\sigma^2}}{\sigma^2} > 0,\ \delta_4 = \frac{0.5\sigma^2+\sqrt{0.25\sigma^4+2r\sigma^2}}{\sigma^2} > 1.
\end{aligned} \tag{A4}$$

We need three conditions to determine three constants. They are value-matching conditions at X_i and X_B,

$$p + A_1X_i^{-\gamma_1} = p + A_2X_i^{-\gamma_2} + A_3X_i^{\delta_2},$$
$$p + A_2X_B^{-\gamma_2} + A_3X_B^{\delta_2} = \frac{X_B}{r},$$

and the smooth-pasting condition at X_i, which gives $-\gamma_1A_1X_i^{-\gamma_1-1} = -\gamma_2A_2X_i^{-\gamma_2-1} + \delta_2A_3X_i^{\delta_2-1}$. Solving these three (linear) equations gives the three constants:

$$A_3 = \frac{X_B/r - p}{\frac{\gamma_1+\delta_2}{\gamma_2-\gamma_1}X_i^{\gamma_2+\delta_2}X_B^{-\gamma_2} + X_B^{\delta_2}},\ A_2 = A_3\frac{\gamma_1+\delta_2}{\gamma_2-\gamma_1}X_i^{\gamma_2+\delta_2}, \text{ and}$$

$$A_1 = A_2 X_i^{\gamma_1-\gamma_2} + A_3 X_i^{\gamma_1+\delta_2}. \tag{A5}$$

Now we move on to equity. First we show the optimality of the threshold investment strategy. Recall the condition $1 > \lambda r$, so that investment is optimal in the first-best scenario. Because at X_B the smooth-pasting condition implies that $E'(X_B) = 0$ while $E'(X) \to \frac{1-\lambda i}{r-i} > \lambda$ as $X \to \infty$ (the first-best level with constant default-free debt value), there must exist a solution to $E'(X_i) = \lambda$. Suppose that we have multiple solutions. Take the smallest one; to prove the optimality of the threshold strategy, it suffices to show that, given the constructed equity value based on the threshold strategy, we must have $E'(X) > \lambda$ for $X > X_i$, where $E(X)$ solves the following ordinary differential equation (ODE) for $X > X_i$:

$$rE(X) = X(1-\lambda i) + iXE'(X) + \frac{1}{2}\sigma^2 X^2 E''(X) - \frac{1}{m}\left(P - D(X)\right), \tag{A6}$$

and $D(X) = p + A_1 X^{-\gamma_1}$ is given in Proposition 2. Suppose not; then there are at least two other solutions $X_1, X_2 > X_i$ so that $E'(X_1) = \lambda$ and $E''(X_1) < 0$; $E'(X_2) = \lambda$ and $E''(X_2) > 0$. We can find some intermediate point $X_3 \in (X_1, X_2)$ satisfying $E'(X_3) \le \lambda$, $E''(X_3) = 0$, and $E'''(X_3) > 0$. Geometrically, X_3 is the bottom point of $E'(X)$, so that the function $E'(X)$ is flat and convex at X_3 . But by taking another derivative of (A5) we have

$$(r-i)E'(X) - 1 + \lambda i = \left(i + \sigma^2\right)XE''(X) + \frac{1}{2}\sigma^2 X^2 E'''(X) + \frac{1}{m}D'(X).$$

Evaluating this equation at X_3, the constant $A_1 < 0$ in equation (A6) implies that $D'(X_3) > 0$. Combining with $E'''(X_3) > 0$, we have the left-hand side $(r-i)E'(X_3) - 1 + \lambda i > 0$. However, from $E'(X_3) \le \lambda$, we have a contradiction (recall $\lambda r < 1$ so that investment is efficient in the first-best scenqrio):

$$(r-i)E'(X_3) - 1 + \lambda i \le (r-i)\lambda - 1 + \lambda i = \lambda r - 1 < 0.$$

Now we derive equity value. Firm value $V(X)$ satisfies

$$V(X) = \begin{cases} \dfrac{X(1-\lambda i)}{r-i} + B_1 X^{-\gamma_3}, & X \ge X_i \\ \frac{X}{r} + B_2 X^{-\gamma_4} + B_3 X^{\delta_4}, & X_B < X < X_i. \end{cases}$$

We solve for the constants B_i based on the value-matching conditions and smooth-pasting conditions:

$$B_3 = \frac{(1+\gamma_3)\dfrac{i(1-\lambda i)}{r(r-i)}X_i^{\gamma_4+1}}{(\gamma_3+\delta_4)X_i^{\delta_4+\gamma_4} - (\gamma_3-\gamma_4)X_B^{\gamma_4+\delta_4}}, \; B_2 = -B_3 X_B^{\gamma_4+\delta_4}, \text{ and}$$

$$B_1 = \frac{i(1-\lambda i)X_i^{1+\gamma_3}}{r(r-i)} + B_2 X_i^{\gamma_3-\gamma_4} + B_3 X_i^{\gamma_3+\delta_4}.$$

We then get equity value $E(X) = V(X) - D(X)$. Q.E.D.

Appendix for Section III.A.4: Based on both $E'(X_i) = \lambda$ and $E'(X_B) = 0$, one can reach the nonlinear equation for $y = X_i / X_B$:

$$1 = \gamma_3 \frac{i}{r} \frac{1 - \delta_4 - (1+\gamma_4) y^{-\gamma_4 - \delta_4}}{\gamma_3 + \delta_4 - (\gamma_3 - \gamma_4) y^{-\gamma_4 - \delta_4}} + \frac{r-i}{1-\lambda r} \frac{\gamma_2 + \delta_2}{\gamma_2 - \gamma_1} \left(\frac{\frac{\gamma_1}{ry} + \frac{\gamma_1 (1+\gamma_3) \frac{i(1-\lambda r)}{r(r-i)} (\gamma_4 + \delta_4)}{(\gamma_3 + \delta_4) y^{\delta_4} - (\gamma_3 - \gamma_4) y^{-\gamma_4}}}{\gamma_2 \frac{\gamma_1 + \delta_2}{\gamma_2 - \gamma_1} y^{\gamma_2} - \delta_2 y^{-\delta_2}} \right). \tag{A7}$$

Now, given the solution y, the default boundary is given by

$$X_B = rp \frac{\gamma_2 \frac{\gamma_1 + \delta_2}{\gamma_2 - \gamma_1} - \delta_2 y^{-\delta_2 - \gamma_2}}{\frac{\gamma_1 + \delta_2}{\gamma_2 - \gamma_1} + y^{-\delta_2 - \gamma_2} + \left(\frac{\gamma_1 + \delta_2}{\gamma_2 - \gamma_1} + y^{-\delta_2 - \gamma_2} \right) \frac{y(1+\gamma_3) \frac{i(1-\lambda r)}{r-i} (\gamma_4 + \delta_4)}{(\gamma_3 + \delta_4) y^{\delta_4} - (\gamma_3 - \gamma_4) y^{-\gamma_4}} + \gamma_2 \frac{\gamma_1 + \delta_2}{\gamma_2 - \gamma_1} - \delta_2 y^{-\delta_2 - \gamma_2}}, \tag{A8}$$

which further gives $X_i = yX_B$.

Appendix for Section III.C.1: With the investment cost specification in equation (13), equity holders' optimal strategy still features two thresholds (X_B, X_i): the firm invests whenever $X_t \geq X_i$, does not invest but keeps servicing the debt when $X_t \in (X_B, X_i)$, and defaults whenever X_t drops to X_B. Given X_B, X_i, the debt valuation remains the same as in Proposition 3, and the equity valuation is given by

$$E(X) = \begin{cases} \frac{X(1-\Delta\lambda i)}{r-i} - \frac{\rho}{r} + C_1 X^{-\gamma_3} - p - A_1 X^{-\gamma_1}, & X \geq X_i \\ \frac{X}{r} + C_2 X^{-\gamma_4} + C_3 X^{\delta_4} - p - A_2 X^{-\gamma_2} - A_3 X^{\delta_2}, & X_B < X < X_i, \end{cases}$$

with

$$C_3 = \frac{(1+\gamma_3) \frac{i(1-\Delta\lambda i)}{r(r-i)} X_i^{\gamma_4+1} - \frac{\Delta\gamma_3}{r}}{(\gamma_3 + \delta_4) X_i^{\delta_4 + \gamma_4} - (\gamma_3 - \gamma_4) X_B^{\gamma_4 + \delta_4}}, C_2 = -C_3 X_B^{\gamma_4 + \delta_4}, \text{ and}$$

$$C_1 = X_i^{\gamma_3} \left[\frac{i(1-\lambda i) X_i^{\gamma_3}}{r(r-i)} + \frac{\rho}{r} + C_2 X_i^{-\gamma_4} + C_3 X_i^{\delta_4} \right].$$

We have the smooth-pasting condition $E'(X_B) = 0$. From $\max_{\tilde{i} \in \{0,i\}} \{ E'(X) \tilde{i} X - \rho - \Delta\lambda \tilde{i} X, 0 \}$, we know that $E'(X_i) = \frac{\rho}{iX_i} + \Delta\lambda$. Recall that $y = X_i / X_B$, and (y, X_B) satisfies the following system of equations:

$$X_B = rp \frac{\gamma_2 \dfrac{\gamma_1+\delta_2}{\gamma_2-\gamma_1} - \delta_2 y^{-\delta_2-\gamma_2}}{\dfrac{\gamma_1+\delta_2}{\gamma_2-\gamma_1} + y^{-\delta_2-\gamma_2} + \left(\dfrac{\gamma_1+\delta_2}{\gamma_2-\gamma_1} + y^{-\delta_2-\gamma_2}\right) \dfrac{\left[y(1+\gamma_3)\dfrac{i(1-\lambda r)}{r-i} - \dfrac{\Delta\gamma_3}{X_B}\right](\gamma_4+\delta_4)}{(\gamma_3+\delta_4)y^{\delta_4} - (\gamma_3-\gamma_4)y^{-\gamma_4}} + \gamma_2 \dfrac{\gamma_1+\delta_2}{\gamma_2-\gamma_1} - \delta_2 y^{-\delta_2-\gamma_2}},$$

$$\begin{aligned} 1 = {} & \gamma_3 \frac{i}{r} \frac{1-\delta_4-(1+\gamma_4)y^{-\gamma_4-\delta_4}}{\gamma_3+\delta_4-(\gamma_3-\gamma_4)y^{-\gamma_4-\delta_4}} \\ & + \frac{r-i}{1-\rho\lambda r} \frac{\gamma_1+\delta_2}{\gamma_2-\gamma_1} \left(\frac{\dfrac{\gamma_1}{ry} + \dfrac{\gamma_1(1+\gamma_3)\dfrac{i(1-\rho\lambda r)}{r(r-i)}(\gamma_4+\delta_4) - \dfrac{\Delta\gamma_3}{X_B}(\gamma_4+\delta_4)}{(\gamma_3+\delta_4)y^{\delta_4}-(\gamma_3-\gamma_4)y^{-\gamma_4}}}{\gamma_2 \dfrac{\gamma_1+\delta_2}{\gamma_2-\gamma_1} y^{\gamma_2} - \delta_2 y^{-\delta_2}} \right) \\ & + \frac{r-i}{1-\rho\lambda r} \left[\frac{\gamma_3}{r} \left(\frac{\delta_4+\gamma_4 y^{-\gamma_4-\delta_4}}{\gamma_3+\delta_4-(\gamma_3-\gamma_4)y^{-\gamma_4-\delta_4}} \right) + \frac{1}{i} \right] \frac{\Delta}{yX_B}. \end{aligned}$$

Proof of Proposition 4: It is easier to work through $f = 1/m \to \infty$. In equation (A4), we see that γ_1, γ_2, and δ_2 are going to infinity at the same order, while other constants are independent of f.

From equation (A7) one can easily show that $y \to 1$ for $f \to \infty$, and y as a function of f is implicitly given by the following equation:

$$y^{\gamma_2(f)} = \frac{r-i}{r(1-r\lambda)} \frac{1+(1+\gamma_3)\dfrac{i(1-r\lambda)}{r-i}}{1+\gamma_3 i/r}, \tag{A9}$$

which is bounded. Hence, $y^{\gamma_2+\delta_2}$ is bounded as well. Based on equation (A8), we have

$$X_B = rp \frac{\gamma_2}{1+(1+\gamma_3)\dfrac{i(1-r\lambda)}{r-i} + \gamma_2}. \tag{A10}$$

This immediately implies that $X_B \to rp$, which is the first claim in Proposition 4.

To show the second claim, we need to calculate the endogenous p that achieves the initial debt value target D_0. Denote the initial assets-in-place by X_0. Using the solution for A_3 in (A5), one can show that $A_3 \to X_B^{\gamma_1}\left(\frac{X_B}{r} - p\right)$, which implies that

$$p_0 = \frac{D_0 - (X_0/X_B)^{-\gamma_1} X_B}{1-(X_0/X_B)^{-\gamma_1}}. \tag{A11}$$

Define $a \equiv 1 + (1+\gamma_3)\frac{i(1-r\lambda)}{r-i} > 0$ and $b \equiv \frac{r-i}{r(1-r\lambda)} \frac{a}{1+\gamma_3 i/r} > 0$. Then, when $f \to \infty$, (A10) and (A9) can be rewritten as $X_B = rp\frac{\gamma_2}{a+\gamma_2}$ and $y^{\gamma_2(f)} = b$. Combining X_B

with (A11) yields $(rD_0 - X_B)\gamma_2 = aX_B\left(1 - (X_0/X_B)^{-\gamma_1}\right)$, which implicitly defines $X_B(f)$. Hence,

$$\frac{\partial X_B}{\partial f} = -\frac{aX_B\gamma'(f)\left[\frac{1}{\gamma} - \frac{1}{\gamma}(X_0/X_B)^{-\gamma} - (X_0/X_B)^{-\gamma}\ln\frac{X_0}{X_B}\right]}{-a + \gamma + a(1+\gamma)(X_0/X_B)^{-\gamma}},$$

where we denote γ_1, γ_2 by γ when $f \to \infty$. In the denominator, the second term dominates. In the numerator inside the square bracket, the second and third terms vanish relative to the first term. Hence, $X_B'(f) \to \frac{aX_B\gamma'(f)}{\gamma^2}$. Besides, $y'(f) \to -\frac{y\ln b}{\gamma^2}\gamma'(f)$. Because $X_i = X_B y$, we have $X_i'(f) = (a - \ln b)X_i\frac{\gamma'(f)}{\gamma^2}$, which implies that X_i increases in f if $a > \ln b$ holds when $f \to \infty$. Finally, notice that when $\lambda \to 0$ so that the investment threshold X_i converges to X_B, one can show that $b = 1$ and hence the condition holds always. Q.E.D.

Appendix for Section III.E: We give the equity and debt value with second stage expansion only. Footnote 18 and $\theta > 1$ imply that $\theta X_B > pr$. Thus, given the expansion opportunity, firm value after expansion is $\theta X_t/r \geq \theta X_B/r$, and debt (equity) holders receive p ($\theta X_t/r - p \geq 0$).

Define the following constants:

$$\gamma_5(m) = \frac{-0.5\sigma^2 + \sqrt{0.25\sigma^4 + 2\sigma^2\left(r + 1/m + \xi\right)}}{\sigma^2} > 0,$$

$$\gamma_6 = \frac{-0.5\sigma^2 + \sqrt{0.25\sigma^4 + 2\sigma^2(r+\xi)}}{\sigma^2} > 0, \tag{A12}$$

with $\gamma_5 > \gamma_6$. Given X_B debt value is $D(X) = p + X_B^{\gamma_5}\left(\frac{X_B}{r} - p\right)X^{-\gamma_5}$, firm value is

$$V(X) = \frac{X}{r}\frac{r+\xi\theta}{r+\xi} - \frac{X_B^{1+\gamma_6}}{r}\frac{(\theta-1)\xi}{r+\xi}X^{-\gamma_6} = \frac{X}{r} + \frac{X}{r}\frac{(\theta-1)\xi}{r+\xi}\left[1 - \left(\frac{X}{X_B}\right)^{1+\gamma_6}\right],$$

and equity value is

$$E(X) = V(X) - D(X) = \frac{X}{r}\frac{r+\xi\theta}{r+\xi} - \frac{X_B^{1+\gamma_6}}{r}\frac{(\theta-1)\xi}{r+\xi}X^{-\gamma_6} - p - X_B^{\gamma_5}\left(\frac{X_B}{r} - p\right)X^{-\gamma_5}.$$

The smooth-pasting condition $E'(X_B) = 0$ implies that

$$X_B = \frac{rp\gamma_5}{\gamma_5 - \gamma_6 + (1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}. \tag{A13}$$

It is clear that X_{B} is increasing in p. For refinancing frequency f (thus maturity m), notice that (A12) suggests that $\gamma_5'(m) < 0$ while γ_6 is independent of m. We thus have

$$X_B'(m) = \frac{\dfrac{rp\gamma_5'(m)}{r+\xi}\left(\gamma_6\xi(\theta-1)+r+\xi\theta\right)}{\left(\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}\right)^2} < 0.$$

Proof of Proposition 5: We work through f instead of $m = 1/f$. Recall that the debt value is

$$D = p + X_B^{\gamma_5}\left(\frac{X_B}{r} - p\right)X^{-\gamma_5} = p + p\left(\frac{rp\gamma_5}{\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}\right)^{\gamma_5} \times \left(\frac{\gamma_6-(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}{\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}\right)X^{-\gamma_5}. \tag{A14}$$

We have

$$\frac{\partial D}{\partial f} = p\left(\frac{-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}{\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}\right)\left(\frac{X_0}{X_B}\right)^{\gamma_5} \tag{A15}$$

$$\times\left[\ln\left(\frac{X_0}{X_B}\right) - \frac{(1+\gamma_6)\dfrac{(\theta-1)\xi}{r+\xi}}{\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}\right], \tag{A16}$$

and

$$\frac{\partial D}{\partial p} = 1 - \left(\frac{X_B}{X_0}\right)^{\gamma_5} - \frac{\gamma_5(1+\gamma_6)\dfrac{(\theta-1)\xi}{r+\xi}}{\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}\left(\frac{X_B}{X_0}\right)^{\gamma_5}. \tag{A17}$$

When X_0 is sufficiently close to X_B, that is, the initial debt value is too high, it is possible that $\frac{\partial D_0}{\partial p} < 0$, that is, the debt value is decreasing in p. Then, by reducing p and thus X_B in (A12), equity holders can find another $\widehat{p}$ so that $\frac{\partial D}{\partial p} > 0$ but still keep the date-0 debt value at D_0. As X_B is lower for $\widehat{p}$, firm value and thus equity value increases. Hence, we know that without loss of generality we can focus on $\frac{\partial D}{\partial p} \geq 0$, that is, (A15) is always positive.

Now we show that X_B in (A13) is increasing in f. We need to check that the following is positive:

$$\frac{dX_B}{df}\Big|_{Keep\,D} = \left[rp\gamma_5'(f) + rp'(f)\gamma_5\right]\left(\gamma_5 - \gamma_6 + (1+\gamma_6)\frac{r+\xi\theta}{r+\xi}\right) - rp\gamma_5'(f)\gamma_5,$$

where $p'(f) = -\frac{\partial D/\partial f}{\partial D/\partial p}$ with partial derivatives given in (A14) and (A15). Rearranging and collecting terms, we arrive at

$$sign\left[\frac{dX_B}{df}\right] = sign\left[\frac{1-\left(\frac{X_B}{X_0}\right)^{\gamma_5} + \gamma_5\left(\frac{X_B}{X_0}\right)^{\gamma_5}\ln\left(\frac{X_B}{X_0}\right)}{1-\left(\frac{X_B}{X_0}\right)^{\gamma_5} - \dfrac{\gamma_5\left(X_B/X_0\right)^{\gamma_5}(1+\gamma_6)\dfrac{\xi(\theta-1)}{r+\xi}}{\gamma_5-\gamma_6+(1+\gamma_6)\dfrac{r+\xi\theta}{r+\xi}}}\right].$$

The denominator, which is (A15), is always positive because of equity holders' optimality condition (see the reasoning right after (A15)). We need to show that the numerator is positive as well. To see this, let $z \equiv \left(X_B/X_0\right)^{\gamma_5} \in (0,1)$ as $X_0 > X_B$. Therefore, the above condition is equivalent to $1 - z + z\ln z > 0$, which holds for $z \in (0,1)$ (to see this, $1 - z + z\ln z$ has a derivative that is negative for $z \in (0,1)$, while it equals zero when $z = 1$). Hence, $\frac{dX_B}{df} > 0$ and thus $\frac{dX_B}{dm} < 0$. Q.E.D.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's web site:

Internet Appendix