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Leverage Dynamics without Commitment

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ABSTRACT

We characterize equilibrium leverage dynamics in a trade-off model in which the firm can continuously adjust leverage and cannot commit to a policy ex ante. While the leverage ratchet effect leads shareholders to issue debt gradually over time, asset growth and debt maturity cause leverage to mean-revert slowly toward a target. Investors anticipate future debt issuance and raise credit spreads, fully offsetting the tax benefits of new debt. Shareholders are therefore indifferent toward the debt maturity structure, even though their choice significantly affects credit spreads, leverage levels, the speed of adjustment, future investment, and growth.

UNDERSTANDING THE DETERMINANTS OF A FIRM'S CAPITAL structure and how its leverage is likely to evolve over time is one of the central questions in corporate finance. Leverage and its expected dynamics are crucial to valuing the firm, assessing its credit risk, and pricing its financial claims. Forecasting the optimal response of leverage to shocks is necessary to anticipate the likely consequences of a crisis and its aftermath, and to evaluate alternative policy responses.

Despite its importance, a fully satisfactory theory of leverage dynamics has yet to be identified. Many models assume that the absolute level of debt is

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[Correction added on 16 January 2021, after first online publication: Minor corrections have been made for clarity in the equations on pages 14, 17, 18, 33, 46, 49 and 50.]

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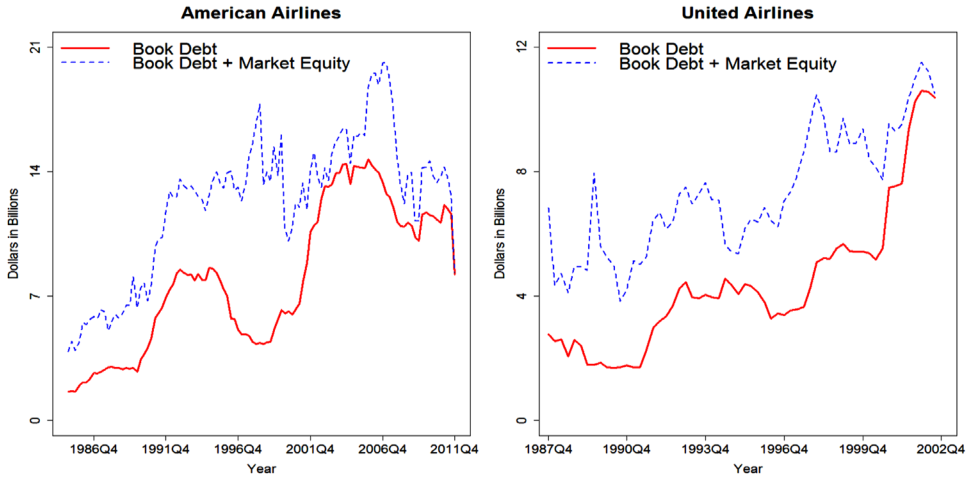

Figure 1. Time series of outstanding book debt and enterprise value for American and United Airlines, for the 15 years before their bankruptcies in 2011 and 2002, respectively. Book debt is calculated as the sum of "long-term debt" and "debt in current liabilities," and market equity is calculated as the product of "stock price" and "common shares outstanding." Data source: WRDS.

fixed. For example, in the traditional framework of Merton (1974), as well as Leland (1994, 1998), the firm commits not to change its outstanding debt before maturity, irrespective of the evolution of the firm's fundamentals. As a result, the dynamics of firm leverage are driven solely by the stochastic growth in the value of the firm's assets-in-place. More recent work that allows the firm to restructure its debt over time typically assumes that all existing debt must be retired (at a cost) before any new debt can be put in place.[1] However, these assumptions are neither innocuous, as the constraints on leverage generally bind in the model, nor consistent with practice, as firms often borrow incrementally over time. See, for example, Figure 1, which shows how debt levels for American and United Airlines changed over time in response to fluctuations in their enterprise values (market value of equity plus book value of debt).

In this paper, we study a model in which equity holders lack the ability to commit to their future leverage choices and can issue or buy back debt at the prevailing market price at any time. Apart from corporate taxes and default costs, there are no other frictions or transaction costs in our model. Because debt can be freely adjusted over time, the firm *could* increase debt to exploit tax shields when cash flows are high and also reduce debt to avoid distress costs when cash flows fall, thereby avoiding the standard leverage "trade-off." However, while such a policy is feasible, it is not time consistent. As Admati et al. (2018) emphasize, the desire of equity holders to maximize the current share price leads to a "leverage ratchet effect," whereby equity holders are

[1] See for example, Fischer, Heinkel, and Zechner (1989), Titman and Tsyplakov (2007), Goldstein, Ju, and Leland (2001), Strebulaev (2007), and Dangl and Zechner (2016).

never willing to voluntarily reduce leverage but always have an incentive to borrow more—even if current leverage is excessive and even if new debt must be junior to existing claims.

We solve for the equilibrium leverage dynamics that emerge when the firm cannot commit to a future leverage policy. We show that as a result of the leverage ratchet effect, (i) the firm issues debt smoothly, and leverage adjusts slowly, in response to changes in firm profitability; (ii) although shareholders never repurchase debt, leverage mean-reverts to a target due to asset growth and debt maturity; (iii) even though the interest tax shield motivates debt issuance, in equilibrium credit spreads rise to the point that the firm is unable to capture these tax shields; and (iv) default costs and investment distortions fully offset the benefits of leverage, so that the firm is indifferent to its future debt maturity structure.

Solving dynamic leverage models is complicated by the interdependence between current debt prices and the firm's future leverage and default decisions. We develop a methodology to characterize the no-commitment equilibrium in a general setting that allows for finite-maturity debt, asset growth, investment, and both Brownian and Poisson shocks. In equilibrium, equity holders issue debt gradually over time, at a rate that depends on the current profitability of the firm. Equity holders find it optimal to issue new debt to exploit tax benefits even after the firm's leverage passes above the "optimal" level with commitment, leading to excessive inefficient default.[2] Firms never actively reduce leverage, but do allow it to decline passively via debt maturity or asset growth. Creditors anticipate the devaluation associated with future over-borrowing and as a result lower the price they are willing to pay for debt today. Higher credit spreads offset the tax advantage of leverage, which slows the rate of issuance to the point that, on the margin, equity holders are indifferent to leverage increases. We show that this effect is so strong that equity holders obtain the same value in equilibrium *as if* they committed not to issue any debt in the future. In other words, the extra tax shield benefits that tempt shareholders are exactly offset by the bankruptcy costs caused by excessive leverage.

The result that all gains from trade are offset in equilibrium is closely related to Coase's (1972) conjecture regarding durable goods monopoly. In that context, the firm is a monopoly supplier of its own debt, and creditors' valuation of the debt depends on the total quantity supplied. In our model, as Coase (1972) conjectured, the equilibrium debt price falls to the marginal (after-tax) cost to shareholders, and thus, shareholders are unable to capture the gains from trade.[3]

[2] This result holds even when there is *no* dilution motive to issue debt (because there is zero recovery value in bankruptcy, or because debt is prioritized so that newly issued debt must be junior to all existing debt). However, even without a direct dilution effect, there is an indirect "dilution" or devaluation effect associated with new debt issuance, as additional leverage increases the probability of default for all debt holders.

[3] Though note that, unlike in Coase's (1972) setting, marginal cost (which is the equilibrium debt price) is endogenous in our model, and investors share a common valuation for the asset.

We apply our methodology to the special case of geometric Brownian motion (as in Leland (1994)) and solve in closed form for the debt price and issuance policy in the unique Markov perfect equilibrium (MPE). Debt accumulates over time at a rate that increases with profitability, but if profits decline sufficiently, new issuance drops below the rate of debt maturity and hence the debt level falls. Leverage is thus path dependent, and the firm's outstanding debt at any point in time can be explicitly expressed in terms of a weighted average of the firm's past earnings. The firm's interest coverage ratio mean-reverts gradually, with the speed of adjustment decreasing with debt maturity and asset volatility. Our model thus provides a theoretical foundation for partial adjustment models (e.g., Jalilvand and Harris (1984), Leary and Roberts (2005), and more recently, Frank and Shen (2019)) that are widely used in the empirical capital structure literature. Importantly, these dynamics differ from the abrupt adjustment to a "target" leverage level implied by models with an exogenous adjustment cost (for instance, Fischer, Heinkel, Zechner (1989), Goldstein, Ju, Leland (2001), and Strebulaev (2007)).

We compare our model without commitment to the benchmark in which shareholders commit to *never* issue debt in the future. We show that equity prices coincide in both settings, whereas bond investors' anticipation of future borrowing leads credit spreads to be much larger in the setting without commitment. For a firm that is initially unlevered, we show that this high cost of debt may dissuade it from borrowing altogether and thus help explain the zero-leverage puzzle (Strebulaev and Yang (2013)).

We also study the optimal debt maturity structure. Our model without commitment provides a fresh perspective on this question. We show that at each point in time, shareholders are indifferent to the maturity choice for future debt issuance. Short-maturity debt leads to higher leverage on average, as shareholders issue debt more aggressively knowing that leverage can be reversed more quickly when the short-term debt matures. Nevertheless, the gain from additional tax shields is offset by increased default costs. Thus, the agency costs associated with the leverage ratchet effect are bounded away from zero even as debt maturity becomes arbitrarily short.

Although firms are indifferent to the debt maturity choice, alternative maturity structures produce widely different leverage dynamics. Thus, small perturbations or frictions that induce firms to select a particular initial maturity structure can lead over time to dramatically different leverage outcomes.[4]

Similar results can be found in DeMarzo and Uroševic (2006) in the context of trading by a large shareholder, and in Daley and Green (2018) where a monopolistic buyer makes frequent offers to a privately informed seller.

[4] The choice of debt maturity structure does affect the value of equity if the firm is forced to borrow a fixed amount up front. This question is studied in the Leland (1998) setting, and long-term debt, which minimizes rollover risk, is often preferred (He and Xiong (2012), Diamond and He (2014)). In contrast, we show that without commitment, firms prefer short-term debt for any positive targeted debt financing. Short-term debt has lower price impact because it allows the firm to reduce leverage quickly through maturity. (Another possible force favoring short-term debt is investors' liquidity preference, which is modeled in He and Milbradt (2014).) On the other hand,

In addition, because leverage mean-reverts slowly, even firms with the same debt maturity may have persistent differences in leverage resulting from past shocks. These results provide a potential explanation for findings such as those by Lemmon, Roberts, and Zender (2008) that much of the cross-sectional variation in firms' capital structure is persistent and largely unexplained by observable firm characteristics.

Finally, we consider the interaction between the firm's leverage and investment policies. When the firm cannot commit to its investment policy, leverage distorts investment due to debt overhang. The firm tends to issue new debt more slowly, and targets a lower level of leverage, in the presence of debt overhang. Near default, however, shareholders raise current cash flows by cutting investment inefficiently and issuing debt more aggressively than if investment were fixed. In equilibrium, underinvestment costs substitute for default costs, again to the point that expected future tax benefits are exactly offset. As a result, the ability to commit to an investment policy does not improve the ex ante value of the firm, in contrast to standard trade-off models.[5] Furthermore, tax policies or other subsidies designed to promote investment by lowering the cost of debt may actually reduce investment in equilibrium (see also Crouzet and Tourre (2020)).

Our paper is most closely related to Admati et al. (2018). They demonstrate the leverage ratchet effect in the context of a one-time leverage adjustment and then numerically evaluate a dynamic equilibrium in a stationary model with regime shocks and perpetual debt. Our paper studies leverage dynamics in a richer continuous-time framework that allows for both asset growth and debt maturity, as well as Brownian and Poisson shocks. As one of the main contributions of the paper, we develop a general methodology to solve for an important class of equilibria. For the standard workhorse model of Leland (1994), we not only solve for the equilibrium in closed form allowing for deeper analysis, but also establish the uniqueness of this equilibrium.

In Dangl and Zechner (2016), the firm can choose how much maturing debt to roll over, but covenants prevent the firm from increasing the total face value of debt outstanding without first repurchasing all existing debt (at par plus a call premium and a proportional transaction cost). Rolling over debt not only maintains the firm's tax shields, as in our model, but also directly dilutes current creditors given their setting with a strictly positive recovery rate and pari passu debt (which we analyze in Section II.D). They show that when debt maturity is long, equity holders will roll over existing debt fully as it comes due, except for when leverage is so low that recapitalization to a higher face value of debt is imminent (in which case, it is not worthwhile to issue debt that is likely to be replaced soon, at a cost). If debt maturity is sufficiently short, however, then when facing high leverage, shareholders may roll over only a portion of

if the firm is not forced to issue debt initially, we show that long-term debt is preferred from a social perspective because the firm will accumulate its debt more slowly, reducing the expected deadweight losses from default.

[5] In other words, solving the commitment problem regarding the firm's leverage policy is *necessary* for the firm to benefit from being able to commit to an investment policy.

the maturing debt so that the total face value of debt gradually declines. This behavior abruptly reverses when the firm approaches default as shareholders maximize dilution (and minimize equity injections) by again rolling over debt fully.[6] Importantly, they show that firm value is not monotonic in debt maturity; depending on the parameters, an interior optimal maturity may exist that trades off the transaction costs of debt rollover (which favors long maturities) with the benefit from debt reductions given high leverage (which favors short maturities).[7] As in our model, the choice of debt maturity becomes an important commitment device that allows for future debt reductions in the face of negative shocks.

Benzoni et al. (2020) study the role of commitment when the firm faces a *fixed* transaction cost when issuing new debt. They consider the class of "*s*-*S*" capital restructuring policies as in Goldstein, Ju, and Leland (2001) and compare outcomes when the firm can commit to the restructuring boundary to those when it cannot. Naturally, commitment leads to a strictly higher equity value. They also argue that when fixed costs are small and the debt maturity is sufficiently short, equity values without commitment to the restructuring boundaries are close to the outcome with commitment.

In the literature of endogenous debt dynamics in the presence of real investment opportunities, Hennessy and Whited (2005) study one-period short-term debt and highlight the importance of a dynamic framework in testing trade-off theories empirically. Abel (2016) considers a continuous-time model with investment in which firms adjust leverage by issuing debt with instantaneous maturity. Abel assumes i.i.d. regime shocks to profitability and shows that in response to a shock, shareholders never reduce the amount of debt and only firms that are borrowing constrained (i.e., firms that have borrowed an amount equal to 100% of firm value) choose to increase debt.

Our results in this paper highlight the fact that some form of commitment is necessary to capture the potential gains from leverage. DeMarzo (2019) extends our model to incorporate collateral and related commitment mechanisms. Collateral is valuable because it resolves the nonexclusivity problem underlying the leverage ratchet effect, making the value of secured debt insensitive to total leverage. Collateral lowers the cost of capital, and the optimal leverage jumps discretely whenever new collateralizable assets are acquired. The model thus reconciles the empirical evidence regarding the persistence and slow adjustment of capital structure with the strong predictive power of collateral. DeMarzo (2019) also discusses the role and importance of other forms of commitment mechanisms, and the resulting ex post rigidity of capital structure, for firm value.

[6] In the extension of our model in which we allow for direct dilution, because there is no constraint on the rate of issuance, we show that the debt issuance rate increases only at the time of default.

[7] The same trade-off would apply in our model if we were to adopt the same assumption on transaction costs. A similar trade-off exists in Brunnermeier and Yogo (2009), who stress the advantage of short-term debt in providing the firm with flexibility to adjust debt quickly in the face of shocks to firm value, while long-term debt is more effective at reducing costs from rollover risk.

Our paper proceeds as follows. In Section I we introduce a general continuous-time model of the firm and develop our methodology for solving for an equilibrium in which shareholders adjust debt continuously. Section II applies our general results to the special case in which cash flows are lognormal with possible jumps and derives a closed-form solution for security prices and debt issuance. We also consider the case of positive recovery and alternative default regimes with pari passu debt. Section III analyzes debt dynamics and shows that the firm gradually adjusts leverage toward a target level. We then evaluate the effect of the firm's choice of debt maturity on the share price and social welfare, and relate our results to empirical puzzles regarding low-leverage firms. Section IV extends the model to include agency costs of investment. Section V concludes.

I. A General Model

We consider a firm whose cash flows follow a general jump-diffusion process that encompasses typical settings used in the literature, and we include both corporate taxes and bankruptcy costs as in standard trade-off models. We depart from the existing literature by assuming that the firm cannot commit to its future capital structure choices, but instead is free to issue or repurchase debt at any time to maximize the current share value. We analyze the equilibrium no-commitment leverage policy in this setting.

In general, the optimal leverage policy depends on equilibrium debt prices, but debt prices depend on the firm's anticipated *future* leverage choices, which determine the likelihood of default. Although this interdependence complicates the determination of an equilibrium, we specify conditions under which we can construct and characterize the equilibrium leverage policy directly, and demonstrate that the rate of debt issuance is determined by the ratio of the tax benefits from new debt to its price sensitivity to new issues. Surprisingly, despite the issuance of new debt to exploit available tax shields, we show that shareholders do not benefit from this activity—the equilibrium share price is the same as if the firm committed not to issue any new debt in the future.

A. The Firm and Its Securities

All agents are risk neutral with an exogenous discount rate of $r > 0$.[8] The firm's assets-in-place generate earnings before interest and taxes (EBIT) at the rate of Y_t, which evolves according to

$$dY_t = \mu\left(Y_t\right)dt + \sigma\left(Y_t\right)dZ_t + \zeta\left(Y_{t-}\right)dN_t, \tag{1}$$

where the drift $\mu(Y_t)$ and the volatility $\sigma(Y_t)$ are general functions that satisfy standard regularity conditions, dZ_t is the increment of a standard Brownian

[8] Alternatively, we can interpret the model as written under a fixed risk-neutral measure that is independent of the firm's capital structure decision.

motion, dN_t is an independent Poisson increment with intensity $\lambda(Y_t) > 0$, and $\zeta(Y_{t-})$ is the jump size given the Poisson event.[9]

Denote by F_t the aggregate face value of outstanding debt. The debt has a constant coupon rate of $c > 0$, so that over $[t, t + dt]$, debt holders receive coupon payments of $cF_t dt$.[10] The firm pays corporate taxes equal to $\pi(Y_t - cF_t)dt$, where $\pi(\cdot)$ is a strictly increasing function of the firm's profit net of interest.[11] Hence, the net after-tax cost to the firm of the marginal coupon payment is only $1 - \pi' < 1$ (with "prime" indicating derivative), reflecting the debt tax shield subsidy.

For simplicity, we assume that debt takes the form of exponentially maturing coupon bonds with a constant amortization rate $\xi > 0$. More specifically, at each instant, there are $\xi F_t dt$ units of required principal repayments from maturing bonds, corresponding to an average bond maturity of $1/\xi$. Debt retirement in this fashion is similar to a sinking fund that continuously buys back debt at par. Thus, combining interest and principal, equity holders are required to pay debt holders a total flow payment of $(c + \xi)F_t\, dt$ to avoid default.

In the main analysis, we assume investors recover zero value from assets-in-place when equity holders default. The key implication of this assumption, which will simplify our analysis, is that *debt seniority becomes irrelevant*. Because there are no claims to divide in default, old debt holders are *not* directly diluted by new debt holders, where "direct dilution" refers to a decrease in the share of any bankruptcy proceeds going to prior creditors.

We make the zero-recovery-value assumption to emphasize that our results are not driven by the direct dilution that arises when issuing pari passu debt (see, for example, Brunnermeier and Oehmke (2013) and Dangl and Zechner (2016)). Instead, there is an "indirect dilution" effect in our model because the value of existing debt is adversely affected by the increased likelihood of default once new debt is issued. This indirect dilution effect is a form of debt overhang—shareholders exercise their default option earlier if the firm is more indebted. This effect arises even if debt is fully prioritized, with new debt always strictly junior to existing debt.[12] Nevertheless, in Section II.D we consider the case with a positive recovery value in which the firm can issue new pari passu debt.

[9] We have simplified the notation by assuming that the jump size $\zeta(Y_{t-})$ conditional on cash flow Y_{t-} is deterministic. We can easily generalize the model to allow for a random jump size $\tilde{\zeta}(Y_{t-})$, as long as the law of $\tilde{\zeta}(Y_{t-})$ depends on Y_{t-} only.

[10] The coupon rate c is exogenously given in our model, so newly issued debt might not be issued at par. In practice, there may be limits/adjustments to the tax deductibility of the coupon if it is far from the par coupon rate. For simplicity, we ignore the tax consequences of non-par debt issuance for this paper.

[11] Throughout, we use the terms "increasing" and "decreasing" in the weak sense, and add "strictly" as appropriate.

[12] Indeed, our qualitative results still hold with a positive recovery rate when new debt must be junior to existing claims. Extending the model in this way adds significant complexity, however, as debt securities issued at different times have distinct prices. In contrast, given zero recovery or pari passu debt, all debt is identical independent of the timing of issuance.

Equity holders control the outstanding debt F_t through an endogenous issuance/repurchase policy $d\Gamma_t$, where Γ_t represents the cumulative debt issuance over time (which is a right-continuous-left-limit process and measurable with respect to the filtration generated by $\{Y_s : 0 \leq s \leq t\}$).[13] Given our debt maturity assumption, the evolution of the outstanding face value of debt F_t is given by

$$dF_t = d\Gamma_t - \xi F_t dt. \tag{2}$$

Thus, the face value of debt will grow if the rate of issuance more than offsets the contractual retirement rate. To highlight the economic forces at play and in contrast to much of the literature, we assume zero transaction costs in issuing or repurchasing debt.[14]

Given equity holders' expected issuance/repurchase policy, debt holders price the newly issued or repurchased debt in a competitive market. Denote by p_t the endogenous debt price per unit of promised face value. The debt price p_t reflects the information available up to date t, including the current debt issuance $d\Gamma_t$, and hence incorporates the price impact of new borrowing. Importantly, in equilibrium, p_t also reflects creditors' expectations regarding future leverage decisions. Thus, over the time interval dt, the net cash flows to equity holders are

$$\left(\underbrace{Y_t}_{\text{operating cash flow}} - \underbrace{\pi(Y_t - cF_t)}_{\text{tax payment}} - \underbrace{(c+\xi)F_t}_{\text{debt interest \& principal}} \right) dt + \underbrace{p_t d\Gamma_t}_{\text{debt issuance/repurchase}}. \tag{3}$$

The firm continues to operate until the operating cash flow Y_t drops sufficiently low, relative to the outstanding debt level F_t, that equity holders find it optimal to default on their contractual obligation to debtholders. As in Leland (1994, 1998), shareholders cannot commit to a certain default policy, but instead default strategically. Again, for now we assume that all investors receive zero cash flows postdefault, and we consider alternative default payoffs in Section II.D.

B. Smooth Equilibrium

We focus on MPE in which the two payoff-relevant state variables are the firm's *exogenous* operating cash flow, Y_t, and the outstanding aggregate face

[13] To rule out Ponzi schemes in which the firm avoids default by perpetually rolling over all debt, we must impose some upper bound $F_t \leq \bar{F}(Y_t)$ on debt, where the bound $\bar{F}(Y_t)$ exceeds the pretax unlevered value of the firm. This constraint will not bind in equilibrium and plays no role in the analysis.

[14] It is common in the dynamic capital structure literature, such as Fischer, Heinkel, and Zechner (1989) and Goldstein, Ju, and Leland (2001), to assume that firms must buy back *all* of their existing debt to issue new debt, and that there is a positive adjustment cost associated with this transaction. This behavior does not correspond to general practice, however, and we eliminate this constraint to highlight equity holders' intrinsic incentives to adjust leverage.

value of debt, F_t, which is an *endogenous* state variable. We analyze the value function $V(Y_t, F_t)$ for equity and the debt price $p(Y_t, F_t)$. Denote by τ_b the equilibrium default time, that is, the first time that the state (Y_t, F_t) falls into the endogenous default region. We assume for now that all investors receive a payoff of zero if the firm defaults.

We begin by analyzing some general properties that must hold for any equilibrium. In particular, we show that shareholders never choose to repurchase debt, even if leverage is excessive from the perspective of total firm value. This result is a consequence of the leverage ratchet effect (Admati et al. (2018)). In addition, the debt price is decreasing in the amount of debt outstanding, so that debt issuance negatively impacts the debt price.

PROPOSITION 1: (Leverage Ratchet and Price Impact)*: In any MPE in* (Y, F)*, the firm never repurchases debt, and thus the issuance policy* Γ_t *is a monotonically increasing process. The equity value function* $V(Y, F)$ *is convex and decreasing in* F*, with the debt price as a subgradient:*

$$p(Y, F) \in -\partial_F V(Y, F)$$

Hence, the debt price is decreasing in F*.*

PROOF: To see the convexity of $V(Y, F)$ in F, note that given any debt level F', equity holders have the option to adjust debt to F by issuing $F - F'$ (buying back if this quantity is negative) at the price of $p(Y, F)$. Therefore, the value of the firm given F' must be at least as high as the value that equity would obtain by changing the debt level to F,

$$V(Y, F') \geq V(Y, F) + p(Y, F)(F - F') = V(Y, F) - p(Y, F)(F' - F), \tag{4}$$

establishing that the (negative) debt price is a subgradient of V. As a result, V is convex in F and debt price is decreasing in F. Because $p \geq 0$, V is also decreasing in F.

To see why shareholders would not benefit from a debt buyback, consider postponing a planned buyback by dt. Shareholders would save the cost of the buyback (the debt price) today, but then continue to pay the *after*-tax coupons and principal until they repurchase the debt at time $t + dt$. Because the debt price today is equal to the present value of the *pre*-tax coupons, principal payments, and future debt price, shareholders profit from the delay by the amount of the tax shield. In addition, if they delay repurchasing the debt, they also maintain the option to default rather than repurchase. See the Appendix for a formal proof. □

Note that the negative impact of debt issuance on the debt price (i.e., the debt is traded at $p(Y, F)$, where F is the post-trade debt obligation) will deter shareholders from issuing a large amount of debt at once. Indeed, if the value function is strictly convex (and hence the debt price is strictly decreasing) in F, it would be optimal for the firm to adjust debt in a continuous manner.

LEMMA 1: *(Continuous Adjustment): If the equity value function $V(Y, F)$ is strictly convex in F, then the debt price is strictly decreasing in F and the optimal issuance policy Γ_t is continuous in t.*

PROOF: With strict concavity, the inequality in (4) becomes strict. Hence, any discrete issuance with $|F - F'| > 0$ would be suboptimal for shareholders. □

This result motivates us to consider a special class of equilibria in which equity holders find it optimal to adjust the firm's outstanding debt smoothly, with $d\Gamma_t = G_t dt$, where G_t specifies the rate of issuance. Hereafter, we refer to this equilibrium as a "smooth" equilibrium, and we refer to G_t as the equity holders' issuance policy (which must be nonnegative by Proposition 1).[15]

DEFINITION: (Smooth Equilibrium): *A smooth equilibrium is an MPE in which the issuance policy is given by $d\Gamma_t = G_t\, dt$ for some adapted process G_t.*

In the remainder of this section, we develop a methodology to construct and characterize smooth equilibria. Later, in Section II, we consider a specific setting and rule out the existence of nonsmooth equilibria.

C. Security Valuation

In a smooth equilibrium, given the debt price $p(Y, F)$, the firm's issuance policy G and default time τ_b maximize the market value of equity,

$$V(Y, F) = \max_{\tau_b, G} E_t \left[\int_t^{\tau_b} e^{-r(s-t)} \left[Y_s - \pi(Y_s - cF_s) - (c + \xi) F_s + G_s p_s \right] ds \Big| Y_t = Y, F_t = F \right]. \quad (5)$$

Because debt holders receive both coupon and principal payments until the firm defaults, and the firm recovery upon default is assumed to be zero, the equilibrium market price of debt must satisfy

$$p(Y, F) = E_t \left[\int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c + \xi)\, ds \Big| Y_t = Y, F_t = F \right], \quad (6)$$

where the expectation in (6) is under the evolution of F implied by G.

C.1. An Optimality Condition

Recall that we are interested in an equilibrium in which there is no commitment by equity holders to future leverage policies. Thus, at any point in time, the issuance policy G_s for $s > t$ has to be optimal in solving the equity holders' instantaneous maximization problem at time s given equity's value function and equilibrium debt prices.

[15] Technically, there is also the possibility that the issuance policy might include a singular component, so that the sample path of Γ is continuous, but not absolutely continuous. We rule out such policies in our uniqueness proof in Section II.

In this section, we consider the necessary and sufficient conditions for the optimality of the debt issuance policy G_t. The Hamilton-Jacobi-Bellman (HJB) equation for equity holders is

$$\underbrace{rV(Y,F)}_{\text{required return}} = \max_G \Bigg[\underbrace{Y-\pi(Y-cF)}_{\text{after-tax cash flow}} - \underbrace{(c+\xi)F}_{\text{coupon \& principal payment}} + \underbrace{Gp(Y,F)}_{\text{new debt issuance}} + \underbrace{(G-\xi F)V_F(Y,F)}_{\text{evolution of } dF}\Bigg]$$
$$\underbrace{+\mu(Y)V_Y(Y,F)+\frac{1}{2}\sigma(Y)^2V_{YY}(Y,F)+\lambda(Y)\left[V(Y+\zeta(Y))-V(Y)\right]}_{\text{evolution of } dY}. \tag{7}$$

In the first line, the objective is linear in G with a coefficient of $p(Y,F)+V_F(Y,F)$, which represents the (endogenous) marginal benefit of the revenue from a debt sale net of the marginal cost of the future debt burden on shareholders. If shareholders find it optimal to adjust debt smoothly, then it must be the case that this coefficient equals zero, or equivalently,[16]

$$p(Y,F) = -V_F(Y,F). \tag{8}$$

C.2. No-Trade Valuation

The first-order condition (8) is a necessary condition for a smooth equilibrium. While straightforward, it has deep implications for the equilibrium value of equity in any smooth equilibrium. Plugging condition (8) into the equity HJB equation (7), we have the following revised HJB equation for equity:

$$rV(Y,F) = Y-\pi(Y-cF)-(c+\xi)F+\mu(Y)V_Y(Y,F)-\xi FV_F(Y,F)$$
$$+\frac{1}{2}\sigma(Y)^2V_{YY}(Y,F)+\lambda(Y)\left[V(Y+\zeta(Y),F)-V(Y,F)\right]. \tag{9}$$

This equation says that in the no-commitment equilibrium, the equity value can be solved *as if* there is no debt adjustment $G\equiv 0$ (except for the natural retirement at rate ξ).

Note that when $G\equiv 0$, the value of equity is independent of the debt price, and thus is the solution to a standard optimal stopping problem in which Y_t follows (1) and $dF_t = -\xi F_t dt$. These problems are well studied in the literature, as are conditions ensuring the uniqueness and smoothness of the solution.[17] We denote the corresponding no-trade equity value function by $V^0(Y,F)$; here,

[16] While (7) implies (8) in the nondefault region, it is also true in default, as for a defaulted firm the debt price $P=0$ and $V(Y,F)=V_F(Y,F)=0$. (We extend the model to the case with positive recovery in Section II.D.) Also, the debt price for $F=0$ is relevant only if the firm were to buy back all of its debt (which is off-equilibrium according to Proposition 1), and hence setting $p(Y,0) \geq -V_F(Y,0)$ is sufficient for optimality.

[17] See, for example, chapter 10 of Oksendal (2013), who analyzes a case with a finite-dimensional diffusion. Ishikawa (2011) offers an analysis covering the case of jump-diffusion processes. We assume that the appropriate technical conditions hold throughout the paper.

"no trade" simply means shareholders do not participate in the debt market. In other words, V^0 satisfies (9) and is the solution to (5) when G is constrained to be zero. The preceding argument then implies that the equity value must equal this no-trade value in any smooth equilibrium.

PROPOSITION 2: (No-Trade Equity Valuation)*: In any smooth equilibrium, the value of equity is equal to the no-trade valuation:* $V(Y,F) = V^0(Y,F)$.

PROOF: Given any smooth equilibrium with value function V, (7) and (8) imply (9), and as a result setting the issuance policy to $G = 0$ does not change the equity value V. Hence, the value V under this equilibrium could be obtained with no trade. But because V^0 is the optimal value with no trade (with a potentially superior default policy), $V(Y,F) \leq V^0(Y,F)$. But because equity holders can always choose not to trade, $V(Y,F) \geq V^0(Y,F)$ for any equilibrium. Hence, we have $V(Y,F) = V^0(Y,F)$. □

Intuitively, because equity holders gain no marginal surplus from adjusting the debt level, their equilibrium payoff must be the same as if they were to never issue/repurchase any debt. This result, while perhaps striking at first, is analogous to the Coase (1972) conjecture for durable goods monopoly—the firm is a monopolist issuer of its own debt. When a monopolist is unable to commit to restricting its future sales, it cannot resist the temptation to sell aggressively, so much so that the endogenous price falls to marginal cost and any surplus from trading gets dissipated in equilibrium.[18]

C.3. Optimal Debt Issuance

Proposition 2 implies that if a smooth equilibrium exists, we may compute the equity value function *as if* there were no further trade and the firm gradually retires its existing debt. Given the equity value V, we can then invoke the first-order condition in (8) to obtain the equilibrium debt price $p(Y,F) = -V_F(Y,F)$.

The question remains, however, as to whether this debt price is consistent with a smooth issuance policy G that is also optimal for shareholders. It is straightforward to show that $G = 0$ cannot be an equilibrium, as in that case the debt price would exceed its marginal cost to shareholders due to the interest tax shield, and so shareholders would find $G > 0$ optimal.[19] But as the rate of issuance increases, the likelihood of default will increase and the price of debt will fall to the point that (8) holds.

[18] A closely related result appears in DeMarzo and Uroševic (2006) in a model of trade by a large shareholder trading off diversification benefits and price impact due to reduced incentives. In equilibrium, share prices are identical to those implied by a model with no trade. Similarly, the monopolist buyer in Daley and Green (2018) that cannot commit to their future strategy gains nothing from their ability to screen buyers over time.

[19] With no future debt issuance, the marginal cost to shareholders of an extra \$1 of debt is the present value of the after-tax debt payments until default (noting that the effect of any change in the timing of default is of second order thanks to the envelope theorem), whereas the debt price is the present value of the pretax payments.

To determine the equilibrium debt issuance policy, which we denote by G^*, we see from (6) that the debt price must satisfy the standard HJB equation,

$$\underbrace{rp(Y,F)}_{\text{required return}} = \underbrace{c}_{\text{coupon}} + \underbrace{\xi(1-p(Y,F))}_{\text{debt retirement}} + \underbrace{(G^*-\xi F)p_F(Y,F)}_{\text{evolution of debt } dF} + \underbrace{\mu(Y)p_Y(Y,F) + \frac{1}{2}\sigma(Y)^2 p_{YY}(Y,F) + \lambda(Y)\left[p(Y+\zeta(Y),F) - p(Y,F)\right]}_{\text{evolution of cash flow } dY}. \tag{10}$$

Next, starting with the HJB equation (9) for $V(Y,F)$, if we differentiate by F and use the optimality condition $p = -V_F$, we obtain

$$\begin{aligned} -rp(Y,F) = {} & \pi'(Y-cF)\cdot c - (c+\xi) + \xi p(Y,F) + \xi F p_F(Y,F) \\ & - \mu(Y)p_Y(Y,F) - \frac{1}{2}\sigma(Y)^2 p_{YY}(Y,F) \\ & + \lambda(Y)\left[-p(Y+\zeta(Y),F) + p(Y,F)\right]. \end{aligned} \tag{11}$$

Although equation (11) is written in terms of the debt price p, we emphasize that it follows mechanically from the equity valuation equation (9), together with the first-order condition (8) for the optimal issuance policy. Finally, adding (11) to (10), we obtain the following result on how to construct a smooth equilibrium and characterize the equilibrium debt issuance policy:

PROPOSITION 3: (Equilibrium Construction and Optimal Issuance)*: Suppose the no-trade value $V^0(Y,F)$ is twice-continuously differentiable and strictly convex in F (outside the default region).*[20] *Then $V = V^0$ is the unique smooth equilibrium value function, with debt issuance policy*

$$G^*(Y,F) = \frac{\pi'(Y-cF)\cdot c}{-p_F(Y,F)} = \frac{\pi'(Y-cF)\cdot c}{V_{FF}(Y,F)}. \tag{12}$$

Under this policy, the debt price given by (6) satisfies $p = -V_F$.

PROOF: Given a smooth policy, (7) and (8) imply (11), which combined with (12) implies that $-V_F$ satisfies (10). Because $p = -V_F = 0$ at the default boundary, we have that $p = -V_F$ satisfies (6). Finally, the global optimality of the issuance policy follows from $p = -V_F$ and the strict concavity of V. □

While this result provides sufficient conditions for the existence of a smooth equilibrium and provides its characterization, it does not rule out nonsmooth

[20] We note that a simple sufficient condition for convexity in F is a constant marginal tax rate (for a more general proof including investment, see the proof of Proposition 11. For the "smoothness" condition of the no-trade value function $V^0(Y,F)$, which is typically twice-continuously differentiable in the interior (and continuously differentiable on the boundary), see discussion in footnote 17.

equilibria. We give conditions for the smooth equilibrium to be the unique MPE in Section II.

We can interpret the equilibrium debt issuance policy (12) as follows. The rate of issuance of debt is such that the rate of devaluation of the debt induced by new issuances just offsets the marginal tax benefit associated with the coupon payments,

$$-G^{*}(Y,F)\ \ p_{F}(Y,F)=\pi'(Y-cF)\ \ c. \tag{13}$$

Note that the debt issuance is strictly positive given the strictly positive marginal tax rate. Again, this result is consistent with the leverage ratchet effect of Admati et al. (2018)—even if the firm's current leverage is excessive, equity holders never actively reduce debt but always have an incentive to increase debt when it provides a marginal tax benefit.

C.4. Discrete Optimization

In equilibrium, because $p=-V_F$, the value of equity is the same for *any* smooth issuance policy G. The equilibrium policy G^* is then uniquely determined so that the resulting debt price makes shareholders indifferent. In a sense, our characterization of G^* is analogous to that of a mixed strategy equilibrium in which each player is indifferent to her choice of action, yet equilibrium strategies are uniquely determined to maintain that indifference.

Shareholder indifference regarding the issuance policy is, however, an artifact of the continuous-time limit. If we were to compute the equilibrium as the limit of a discrete-time model, the optimal policy G^* would arise as the result of a strict optimization by shareholders. To see this result heuristically, suppose that the firm issues debt Δ, which is fixed over the next instant dt, and let p and V be the end-of-period debt price and equity value functions, respectively. The firm would then pay additional interest of $\Delta c dt$, and thus its earnings would decline by $(1-\pi'(Y-cF))\Delta c dt$ on an after-tax basis.[21] Because the bonds trade for a cum-coupon price of $cdt+p(Y,F+\Delta)$, shareholders would choose Δ to solve[22]

$$\max_{\Delta} \underbrace{-\left(1-\pi'(Y-cF)\right)\Delta c dt}_{\text{after}\,-\,\text{tax interest payment}}+\underbrace{\Delta\left[cdt+p(Y,F+\Delta)\right]}_{\text{debt proceeds}}+V(Y,F+\Delta)\,. \tag{14}$$

[21] Here we are ignoring terms of order dt^2 or higher, which would arise if the marginal tax rate is not locally constant.

[22] Recall that p is the end-of-period bond price. If sold earlier it will trade for a higher price that includes the initial coupons. Also, we assume that the new debt issuance occurs after the current period's default decision and principal repayments; changing the timing would introduce terms of order dt^2 without altering the conclusion.

This maximization has the first-order condition

$$\Delta = \frac{\pi'(Y - cF)cdt + \overbrace{(p + V_F)}^{\substack{\text{zero by equilibrium} \\ \text{condition (8)}}}}{-p_F} = \frac{\pi'(Y - cF)c}{-p_F}dt, \tag{15}$$

which exactly coincides with (12). Hence, we can interpret G^* as the strictly optimal issuance rate when the firm has "infinitesimal" commitment power over $[t, t + dt]$ in a discrete-time setting.[23]

C.5. Summary

In sum, for the general model without commitment in which equity holders are free to issue or repurchase any amount of debt at the prevailing market price, one can solve for the smooth equilibrium as follows:[24]

(i) Use (9) to solve for the equity holder's value function $V(Y, F) = V^0(Y, F)$ by setting $G = 0$ (i.e., as if equity holders commit not to issue any future debt).
(ii) Set the debt price $p(Y, F) = -V_F(Y, F)$.
(iii) Check for global optimality by verifying that, up to the point of default, the debt price $p(Y, F)$ is strictly decreasing in F (equivalently, V is strictly convex in F).
(iv) Given $p(Y, F)$, solve for the equilibrium issuance policy $G^*(Y, F)$ from (12).

In the remainder of the paper we use this methodology to analyze several standard settings and consider the consequence for debt valuation and leverage dynamics. We also show that we can rule out any nonsmooth Markov equilibria.

II. An Explicit Solution

We now apply the general methodology developed in the previous section to the widely used framework of a lognormal cash flow process and derive an explicit solution.[25] The results from Section I allow us to fully characterize an equilibrium in closed form and evaluate the corresponding leverage dynamics. We further extend the model to allow for jumps to cash flows, show that

[23] See also DeMarzo (2019) and DeMarzo, He, and Tourre (2021) for additional discussion of the convergence from discrete to continuous time. Intuitively, if $p + V_F$ is $o(dt)$, then the per-period gain from trade in (14) is $o(dt^2)$, and thus the gains do not aggregate in the limit.

[24] This general approach also applies to the case in which shareholders make endogenous investment decisions along with leverage decisions, as shown in Section IV and explored further in the NBER working paper version https://www.nber.org/papers/w22799

[25] This setting is consistent with, for instance, Merton (1974), Fischer, Heinkel, and Zechner (1989), Leland (1994), and Leland and Toft (1996), and follows the development of starting from cash flows rather than firm value as in Goldstein, Ju, and Leland (2001).

the solution is qualitatively unchanged, and establish the uniqueness of the Markov equilibrium in this setting. Finally, in Section II.D we study the case of a positive recovery value; shareholders' ability to restructure the debt to their advantage using the threat of dilution makes the debt price more sensitive to new issuance, thereby reducing the equilibrium level of debt.

A. Lognormal Cash Flows

In the special case of lognormal operating cash flows, Y_t follows a geometric Brownian motion,

$$\mu\left(Y_t\right) = \mu Y_t \text{ and } \sigma\left(Y_t\right) = \sigma Y_t, \text{ with } r > \mu. \tag{16}$$

To maintain homogeneity, we also assume a constant tax rate $\bar{\pi} > 0$, so that[26]

$$\pi(Y_t - cF_t) = \bar{\pi} \cdot (Y_t - cF_t). \tag{17}$$

Given the scale invariance of the firm in this setting, the economically relevant measure of leverage is operating cash flow scaled by the outstanding face value of debt,

$$y_t \equiv Y_t / F_t, \tag{18}$$

which is proportional to the firm's *interest coverage ratio*—that is, the ratio of operating income Y_t to total interest expense cF_t—a widely used measure of leverage and financial soundness. (An alternative, equivalent characterization is given in terms of the debt-to-income ratio $f_t \equiv F_t / Y_t = 1/y_t$.) Because all subgames with the same initial leverage y_t are strategically equivalent, we look for an MPE in this unidimensional state variable and show that it must be a smooth equilibrium.[27] That is, given this restriction, the smooth equilibrium is the unique MPE.

In this setting with scale invariance, the equity value function $V(Y, F)$ and debt price $p(Y, F)$ satisfy

$$V\left(Y, F\right) = V\left(\frac{Y}{F}, 1\right) F \equiv v\left(y\right) F \text{ and } p\left(Y, F\right) = p\left(\frac{Y}{F}, 1\right) \equiv p\left(y\right). \tag{19}$$

[26] As with the existing literature, our model adopts an idealized version of the tax code. In practice, the debt tax shield is not strictly tied to the coupon rate, but includes an adjustment for any discount or premium at the time of issuance. In addition, tax shields may be deferred when earnings are negative.

[27] Here we follow Maskin and Tirole (2001), who argue for defining MPE in terms of the coarsest partition such that equivalent subgames are "strategically equivalent." A sufficient condition for strategic equivalence is that the payoffs are equivalent up to an affine transformation (as is the case here).

We solve for the (scaled) equity value function $v(y)$ and debt price $p(y)$ in closed form.

Given the evolution of our state variables Y_t and F_t,

$$dY_t = \mu Y_t dt + \sigma Y_t dZ_t, \text{ and } dF_t = \left(G_t - \xi F_t\right) dt, \tag{20}$$

the scaled cash flows evolve according to,

$$\frac{dy_t}{y_t} = (\mu + \xi - g_t)\, dt + \sigma dZ_t, \text{ where } g_t \equiv G_t / F_t. \tag{21}$$

As (21) shows, because debt F_t grows in a locally deterministic way, the scaled cash flow grows with the same volatility as the total cash flow. The growth rate, however, is increased by the debt amortization rate, ξ, net of the endogenous issuance rate, $g_t \equiv G_t/F_t$. The higher the rate of debt issuance, the slower the growth rate of the scaled cash flow.

When the scaled cash flow y_t falls below some endogenous default boundary y_b, equity holders are no longer willing to service the debt, and therefore choose to strategically default. In that event, we assume for now that both equity and debt holders receive zero liquidation value.

B. Model Solution

Recall from Section I that we can solve for the equilibrium equity value *as if* $g_t = 0$ and hence equity holders do not actively adjust the firm's debt, even though they will do so in equilibrium. Using the fact that

$$V_Y\left(Y, F\right) = v'\left(y\right), \; V_F\left(Y, F\right) = v\left(y\right) - yv'\left(y\right), \text{ and } FV_{YY} = v''\left(y\right), \tag{22}$$

we can rewrite (9) with lognormal cash flows in terms of scaled cash flow y as follows:

$$(r + \xi)\, v\left(y\right) = (y - c - \xi) - \bar{\pi}(y - c) + (\mu + \xi)\, yv'\left(y\right) + \frac{1}{2}\sigma^2 y^2 v''\left(y\right). \tag{23}$$

Note that if the firm could not default, then the cash flows and debt payments could be evaluated as growing perpetuities. Thus, the no-default value of equity would be

$$\bar{v}(y) \equiv \underbrace{\frac{1-\bar{\pi}}{r-\mu}y}_{\text{unlevered asset value}} + \underbrace{\frac{\bar{\pi}c}{r+\xi}}_{\text{tax shield}} - \underbrace{\frac{c+\xi}{r+\xi}}_{\text{bond value}} \equiv \phi y - \rho, \tag{24}$$

where $\phi \equiv \frac{1-\bar{\pi}}{r-\mu}$ is the unlevered valuation multiple for the firm, and $\rho \equiv \frac{c(1-\bar{\pi})+\xi}{r+\xi}$ is the after-tax cost to the firm of a riskless bond.[28] To compute the no-trade equity value, we must add to $\bar{v}$ the value of the default option. The next result characterizes the resulting value function and establishes that this equilibrium is unique (smooth or otherwise).

PROPOSITION 4: (Equilibrium with Lognormal Cash Flows)*: Given lognormal cash flows with constant tax rate $\bar{\pi}$, let*

$$\gamma \equiv \frac{(\mu + \xi - 0.5\sigma^2) + \sqrt{(\mu + \xi - 0.5\sigma^2)^2 + 2\sigma^2 (r + \xi)}}{\sigma^2} > 0. \tag{25}$$

Then the unique MPE in y is the smooth equilibrium, with the equity value function and optimal default boundary given by

$$v(y) = \underbrace{\phi y - \rho}_{\text{no default value}} \underbrace{- \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho)}_{\text{default option value}>0} \quad \text{and} \quad y_b = \frac{\gamma}{1+\gamma}\frac{\rho}{\phi}. \tag{26}$$

PROOF: Given shareholders' option to default and receive zero, the no-trade value function equals

$$v(y) = \bar{v}(y) + \underbrace{E^0\left[e^{-(r+\xi)\tau_b}\right]\left(0 - \bar{v}(y_b)\right)}_{\text{default option value}}, \tag{27}$$

where E^0 denotes the expectation given $g \equiv 0$. The discount factor $h(y) \equiv E^0[e^{-(r+\xi)\tau_b}]$ must solve the homogeneous version of (23),

$$(r + \xi)h(y) = (\mu + \xi)yh'(y) + \frac{1}{2}\sigma^2 y^2 h''(y),$$

with boundary conditions $h(y_b) = 1$ and $h(\infty) = 0$, which is solved by $h(y) = (y/y_b)^{-\gamma}$. The optimal default boundary y_b maximizes the value of the default option and is determined by smooth-pasting, $v'(y_b) = 0$. Because v is smooth and strictly convex (prior to default), Proposition 3 implies that v is the equity value function in the smooth equilibrium with trade. We prove in the Appendix that there does not exist any nonsmooth MPE in y. □

Proposition 4 establishes the unique equilibrium value function within the class of MPE given the firm's interest coverage ratio (or debt-to-income ratio) measured by y. While it is possible to construct other, nonsmooth equilibria if we allow the firm and investors to condition on nonstrategically relevant

[28] The value $\bar{v}$ is a particular solution to (23) that ignores the default boundary condition.

variables (such as the firm's absolute size or age), our equilibrium conforms with standard metrics used in practice to evaluate leverage and credit spreads.[29]

Having solved for the value of equity, recall that we can determine the debt price from the first-order conditions (8). Using (22) and (26), we have

$$p(y) = -V_F = yv'(y) - v(y) = \rho\left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right). \tag{28}$$

Note that the debt price is strictly increasing in y, and therefore strictly decreasing in F, as required in Proposition 3, which we can now apply to (12) to derive the equilibrium debt issuance policy. As shown in the next proposition, the rate of debt issuance $g^*(y)$ is strictly positive and increasing in the scaled cash flow y.[30]

PROPOSITION 5: (Equilibrium Debt Issuance)*: Given lognormal cash flows with constant tax rate* $\bar{\pi}$*, the no commitment debt price is given by (28), and the equilibrium rate of debt issuance is*

$$g^*(y) = \frac{G^*}{F} = \frac{\bar{\pi}c}{-Fp_F(Y,F)} = \frac{\bar{\pi}c}{yp'(y)} = \frac{\bar{\pi}c}{y^2v''(y)} = \frac{\bar{\pi}c}{\rho\gamma}\left(\frac{y}{y_b}\right)^{\gamma}. \tag{29}$$

Thus, with lognormal cash flows, we can fully characterize equilibrium debt dynamics and security pricing in closed form. The equity value equals the value without future trade, implying that shareholders do not benefit from their ability to issue debt in the future. Without commitment, creditors anticipate future debt issuance at the rate given by (29), which depresses the current debt price. Indeed, as the following result demonstrates, the debt price falls by the value of the interest tax shield.

PROPOSITION 6: (Commitment vs. No Commitment Debt Price)*: Let* $p^0(y)$ *be the debt price if the firm committed not to issue future debt* ($g = 0$)*. Then*

$$p^0(y) = p(y) + \underbrace{\frac{\bar{\pi}c}{r+\xi}\left[1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right]}_{\text{debt tax shield value}}. \tag{30}$$

[29] Including additional state variables allows investors to "punish" the firm discontinuously—via a discrete jump in credit spreads that depends on variables other than leverage—for even minor deviations from a proposed equilibrium path. Restricting debt prices to be continuous in firm leverage would be an alternative means to rule out such equilibria. See Maskin and Tirole (2001) for a formalization of the idea that MPE embody the principle that "minor causes should have minor effects." (See also footnote 27 as well as our concluding comments.)

[30] Note that g^* represents the issuance rate as a proportion of the current debt level F, that is, total issuance is $G^* = Fg^*(y)$. Although $g^*(y) \to \infty$ as $F \to 0$, in Section III.A we derive the debt dynamics explicitly starting from $F = 0$ and show that under the optimal policy, the firm's outstanding debt follows a continuous sample path with no jumps.

PROOF: Because the equity value is unchanged, so is the default boundary y_b. Thus,

$$p^0(y) = \underbrace{\frac{c+\xi}{r+\xi}}_{\text{default-free value}} - \underbrace{E^0\left[e^{-(r+\xi)\tau_b}\right]\frac{c+\xi}{r+\xi}}_{\text{expected loss in default}} = \frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b}\right)^{-\gamma}\right).$$

The result follows from comparison with (28). □

Based on the equilibrium values for both equity and debt, total firm value (or total enterprise value, TEV) can be expressed as a multiple of the firm's cash flow (i.e., TEV to EBIT),

$$\frac{V(Y,F)+p(Y,F)F}{Y} = \frac{v(y)+p(y)}{y} = v'(y) = \phi\left[1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right], \tag{31}$$

where the second equality follows from the equilibrium condition for the debt price, and the last equality uses the solution for y_b from (26).

The firm's TEV multiple is strictly increasing in the scaled cash flow y, and thus total firm value *decreases* with leverage. Although there are tax benefits associated with debt, the firm issues debt aggressively enough that the cost of debt rises to offset the tax benefits. In equilibrium, the tax benefits of leverage are fully dissipated by the increase in expected default costs due to continued borrowing. This result is in stark contrast to standard trade-off theory models, such as Leland (1994), in which it is optimal for the firm to issue a large block of debt immediately. In these models, the firm is able to capture a tax benefit only because of its assumed commitment not to issue additional debt.

C. Cash Flow Jumps

In the no-commitment equilibrium, the firm's debt level evolves continuously according to (29). Ostensibly, the smooth issuance policy might seem to depend on the continuity of cash flows and asset values in the diffusion setting. In this section, however, we extend our model to allow the firm's cash flows to jump discontinuously, for example, in response to new product development, and show that our prior solution, whereby shareholders issue debt smoothly, is essentially unchanged.

Consider a jump-diffusion model in which cash flows occasionally jump from Y_t to θY_t for some constant $\theta > 1$.[31] Specifically,

$$dY_t = \hat{\mu} Y_t dt + \sigma Y_t dZ_t + (\theta - 1)Y_{t-}dN_t, \tag{32}$$

[31] While we focus on upward jumps with a fixed size, allowing the upward jump to be stochastic is straightforward. Downward jumps ($\theta < 1$) introduce an extra complication due to jump-triggered default, in addition to diffusion-triggered default. The analysis in Chen and Kou (2009) suggests that one can still solve for the equity valuation in closed form under certain conditions. In any case, our qualitative results hold as long as the equity value function remains convex.

where $\hat{\mu}$ is a constant and dN_t is a Poisson process with constant intensity $\lambda > 0$. In this extension, due to upward jumps, the effective expected asset growth rate becomes

$$\mu \equiv \hat{\mu} + \lambda(\theta - 1). \tag{33}$$

We continue to assume $\mu < r$ to ensure that the unlevered firm value is bounded. In our later comparative static analysis we treat (μ, λ, σ) as primitives (with $\hat{\mu}$ defined implicitly from (33)).

As before, we can solve for the equity value as if shareholders commit not to issue any new debt. Because (32) still maintains scale invariance, $V(Y, F) = F \cdot v(y)$ continues to hold, and the HJB equation for the equity value becomes

$$(r + \xi) v(y) = (1 - \bar{\pi})(y - c) - \xi + (\hat{\mu} + \xi) y v'(y) + \frac{1}{2}\sigma^2 y^2 v''(y) + \lambda \left(v(\theta y) - v(y) \right). \tag{34}$$

The last term in equation (34) captures upward jumps. The usual boundary conditions apply—when $y \to \infty$ and thus leverage is negligible, default risk disappears and $v(y) \to \bar{v}(y)$, while at the point of default, we have value-matching $v(y_b) = 0$ and smooth-pasting $v'(y_b) = 0$.

Our next result shows that even with jumps, equilibrium security prices and debt dynamics have exactly the same functional form as we derived in the diffusion-only case, and that this smooth equilibrium is the unique MPE in state variable y.

PROPOSITION 7: (Equilibrium with Cash Flow Jumps)*: Suppose cash flows evolve as a lognormal diffusion with upward jumps as in (32), with* $\hat{\mu} = \mu - \lambda(\theta - 1)$ *from (33). Then there exists a unique MPE in y, for which the equity value, debt price, and issuance policy are given by (26), (28), and (29), respectively, with* γ *the unique positive root of*

$$W(\gamma) = \lambda\theta^{-\gamma} + \frac{1}{2}\sigma^2\gamma^2 - \left(\mu - \lambda(\theta - 1) + \xi - \frac{1}{2}\sigma^2\right)\gamma - (r + \xi + \lambda) = 0. \tag{35}$$

Given parameters $(\mu, \xi, r, \sigma, \lambda, \theta)$*, the solution* γ *is increasing in* (μ, ξ, r) *and decreasing in* $(\sigma, \lambda, \theta)$*. Holding the growth rate* μ *fixed, the reduction in* γ *from an increase in* $(\sigma, \lambda, \theta)$ *raises the value of equity via the default option.*

PROOF: See the Appendix. □

Thus, an increase in jump frequency or size changes the value of equity similarly to an increase in volatility. Surprisingly, however, even if the firm's profitability Y_t jumps discretely, the equilibrium debt issuance policy (29) continues to hold and debt issuance remains smooth (i.e., of order dt). Thus, in response to a jump in profits, leverage falls discretely before gradually mean-reverting

due to a higher speed of issuance. This property holds even if $\sigma^2 \to 0$ so that the firm's cash flows grow *only* with discrete jumps.[32]

D. Positive Recovery with Pari Passu Debt

Thus far we have assumed that in the event of default, the liquidation value of the firm is zero. Under this assumption, there is no difference between junior and senior debt, which rules out any direct dilution motive for issuing debt. Existing creditors are nonetheless harmed by the issuance of new debt due to its effect on the timing of default.

In this section we restrict attention to pari passu debt and consider the possibility that the firm may have a strictly positive liquidation value in default.[33] While the firm's current creditors might hope to recover these liquidation proceeds, shareholders have an incentive to dilute their claim by issuing new debt prior to default. Indeed, given current debt F, shareholders could capture the fraction $\Delta/(F+\Delta)$ of any recovery value by issuing new debt with a face value of Δ and equal priority just prior to default and then using the proceeds to pay a dividend. Thus, by issuing an arbitrarily large block of debt just prior to default, shareholders could obtain the entire liquidation value, leaving existing creditors with zero recovery if they have no ability to restrict issuance.[34]

In practice, creditors will naturally try to block such extreme dilution by attempting to seize assets, block shareholder payouts, and disrupt operations. Shareholders, in turn, may choose to liquidate assets to fund ongoing operations, or engage in asset substitution, to gamble for resurrection. Shareholder-creditor conflicts throughout this process are likely to sacrifice efficiency and reduce the ultimate recovery value that can be achieved.[35] Often, the resolution of default or distress is a restructuring in which both creditors and shareholders retain some value. Rather than model this complexity directly, in this section we adopt a stylized reduced-form approach that can be calibrated to empirical data to allow for a trade-off between creditor recovery and efficiency.

Suppose shareholders may adopt alternative bankruptcy or restructuring regimes indexed by $j \in J$ that differ in terms of creditors' expected recovery rate, $\beta_j \geq 0$, and expected efficiency, measured by the fraction of the firm's unlevered value $\alpha_j \geq 0$ that is preserved. That is, in regime j, creditors receive a

[32] When $\sigma^2 = 0$, we assume $\hat{\mu} + \xi < 0$, so that, between jumps, cash flows decline faster than debt matures. Absent this assumption, the firm can sustain 100% debt financing without risking default, and the first-best can be obtained. See DeMarzo (2019) for further details.

[33] See DeMarzo (2019) for an analysis including senior secured debt. In that case, the firm can capture tax benefits associated with its collateralized debt.

[34] In other words, there would be a complete violation of absolute priority so that equity holders receive the entire recovery value of the firm (while debt holders recover nothing). Moreover, this dilution can be accomplished at the moment of default—there is no need for the firm to issue debt for the purpose of dilution beforehand. In contrast, Dangl and Zechner (2016) consider pari passu debt with positive recovery, but constrain the rate of debt issuance. Because of this constraint, shareholders issue debt at the maximum speed possible for some period prior to default.

[35] In Section IV we consider endogenous investment/disinvestment, which introduces an additional source of shareholder-creditor conflict.

total expected payoff of $\beta_j \rho F$, while shareholders capture the expected residual value of the firm net of the creditors payoff, $\alpha_j \phi Y - \beta_j \rho F$.[36] We assume that one possible outcome (regime 0) is that, as in Section II.A, shareholders and creditors refuse to agree until all surplus is destroyed ($\alpha_0 = \beta_0 = 0$).[37] Alternative regimes—such as renegotiating or restructuring the debt prior to default, or Chapter 11 versus Chapter 7 bankruptcy proceedings—may differ in terms of their efficiency and degree of creditor protections. More generally, the parameters (α, β) capture in reduced form the consequences of some subgame in which shareholders and creditors adapt their behavior in response to leverage (in Section III we endogenize these parameters in a setting with underinvestment due to debt overhang; we could also consider a setting in which shareholders inefficiently continue the firm and gamble for resurrection).

Given the available alternatives, shareholders choose the default or restructuring regime to maximize their payoff,

$$v^B(y) \equiv \max_{j \in J} \ \alpha_j \phi y - \beta_j \rho. \tag{36}$$

Compared to our initial setting, shareholders now choose both the timing and the mode of default. The existence of restructuring regimes that retain some firm value provides shareholders with an additional strategic option, and this potentially enhances shareholder value and changes the cash flow threshold at which default will occur.

PROPOSITION 8: (Equilibrium with Recovery): *In the unique MPE in y, shareholders will choose the default regime j^* that solves*

$$\Lambda_J = \max_{j \in J} \frac{(1-\beta_j)^{\gamma+1}}{(1-\alpha_j)^{\gamma}} \geq 1. \tag{37}$$

The equity value function and optimal default boundary are

$$v^J(y) = \phi y - \rho \underbrace{-\Lambda_J \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho)}_{\text{default option value}} \text{ and } y_b^J = \frac{1-\beta_{j^*}}{1-\alpha_{j^*}} y_b \geq y_b. \tag{38}$$

The equilibrium debt price and rate of debt issuance are given by

$$p^J(y) = \rho \left(1 - \Lambda_J \left(\frac{y}{y_b}\right)^{-\gamma}\right) \text{ and } g^*(y) = \frac{\bar{\pi} c}{\Lambda_J \rho \gamma} \left(\frac{y}{y_b}\right)^{\gamma}. \tag{39}$$

[36] We normalize the face value of debt by $\rho \equiv \frac{c(1-\bar{\pi})+\xi}{r+\xi}$ to account for the fact that the initial debt price (even with zero leverage) may differ from par. This normalization simplifies expressions but is not otherwise consequential.

[37] We have already seen that by threatening to dilute existing creditors, shareholders can drive their recovery rate β_0 down to zero. The assumption $\alpha_0 = 0$ presumes that creditors can also block payouts to shareholders. When $\alpha_j > 0$, shareholders may receive something from the restructuring.

PROOF: See the Appendix. □

Proposition 8 demonstrates the trade-off between efficiency and debt recovery in restructuring as a function of the parameter γ, which depends on the firm's volatility, growth rate, and debt maturity according to (25) (or, in the case with jumps, (35)). The endogenous parameter $\Lambda_J \geq 1$, which takes the value of one for the baseline case $\alpha_0 = \beta_0 = 0$, fully characterizes the effect of this decision on the equity and debt values. The existence of restructuring regimes with $\Lambda_J > 1$ increases efficiency and raises the value of equity. However, the threat of dilution makes the debt price (and total firm value) more sensitive to leverage, causing the equilibrium rate of debt issuance to decline proportionally with Λ_J.

The option to restructure will raise the default threshold to $y_b^J \geq y_b$. If y_b^J is high enough, the firm may restructure even before cash flows become negative.[38] In that case, in equilibrium the firm never requires access to equity capital (as it does in the standard Leland (1994) model).

III. Debt Dynamics

Now that we have solved for the equilibrium debt issuance policy and security pricing, we can analyze the implications for observed debt dynamics. Although lack of commitment leads the firm to always have a positive rate of debt issuance, the countervailing effects of debt maturity and asset growth cause leverage to mean-revert gradually toward a target. We begin by characterizing this target as well as the speed of adjustment. We then consider the implications of alternative debt maturities. While the firm's target leverage and rate of adjustment are greatly affected by maturity, we show that shareholders are indifferent to any maturity structure for future debt issuance. Thus, similar firms that are both maximizing shareholder value may nonetheless display very different debt dynamics. We conclude by considering welfare implications of debt maturity, as well as several low-leverage puzzles.

A. Target Leverage and Adjustment Speed

As shown in Section II, the equilibrium debt issuance rate $g^*(y)$ is faster when cash flows are high, and slows as the firm approaches default. Because the mapping is monotonic, there is a unique level of scaled cash flow y_ξ^* such that the equilibrium issuance rate will equal the rate of debt maturity,

$$g^*(y_\xi^*) \equiv \xi. \tag{40}$$

[38] The firm will have positive earnings at the time of restructuring if $\frac{1-\beta_{j^*}}{1-\alpha_{j^*}} \geq (\frac{1+\gamma}{\gamma})(\frac{r+\xi}{r-\mu})$. Even if this condition is not met, the firm may have positive cash flows once debt proceeds $g^*(y_b^J)\beta_{j^*}\rho$ are included.

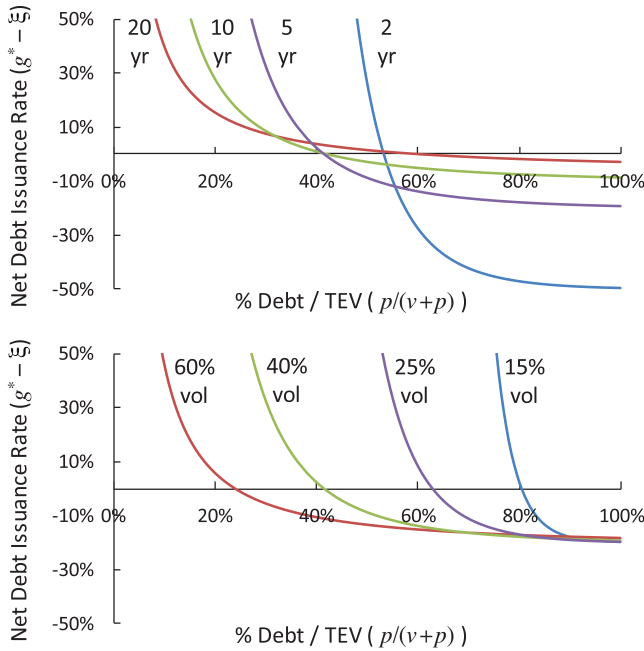

Figure 2. Net debt issuance versus firm leverage for different maturities and volatilities. Baseline parameters: $\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, $c(1-\bar{\pi}) = r = 5\%$, $\xi = 20\%$, $\lambda = 0$, and $\Lambda_J = 1$.

We can also interpret $f_\xi^* \equiv 1/y_\xi^*$ as the firm's leverage "target," the ratio of debt to earnings at which new issuance exactly balances the retirement of existing debt, leaving the firm's total indebtedness unchanged. Over time, leverage will mean-revert toward this target level.[39]

Figure 2 illustrates the net rate of debt issuance, given different debt maturities and asset volatilities, as a function of the firm's debt-to-value ratio. Shorter debt maturity increases the speed of mean reversion, but has a nonmonotonic impact on the target level of leverage. Lower volatility, in contrast, raises both the target level of leverage and the speed of adjustment.

In our model the firm's debt is path dependent, with the current level of debt equal to the firm's cumulative past issuance net of its debt retirement. Because the issuance rate varies with the level of cash flows, this path dependence can be quite complex. Somewhat surprisingly, we can derive the evolution of the firm's debt explicitly as a function of the firm's initial debt position and its earnings history, as shown next.

PROPOSITION 9: (Debt Evolution): *Let $f_\xi^* \equiv 1/y_\xi^*$ be the firm's target ratio of debt to earnings. Given the debt issuance policy g^* and any initial debt face*

[39] Note that if the debt is perpetual or its maturity is very long so that $\xi \approx 0$, then $y_\xi^* < y_b$. In that case the firm's total indebtedness F_t will strictly increase over time (prior to default).

value $F_0 \geq 0$, the firm's debt on date t given the cash-flow history $\{Y_s : 0 < s < t\}$ is

$$F_t = \left[{F_0}^\gamma e^{-\gamma \xi t} + \gamma \xi \int_0^t e^{\gamma \xi (s-t)} \left(f_\xi^* Y_s \right)^\gamma ds \right]^{1/\gamma}. \tag{41}$$

Equivalently, for $dt \approx 0$,

$$F_{t+dt}^\gamma = (f_\xi^* Y_t)^\gamma (\gamma \xi dt) + F_t^\gamma (1 - \gamma \xi dt). \tag{42}$$

PROOF: Recall from (39) and (40) that

$$g^*(y) = \frac{\bar{\pi} c}{\Lambda_J \rho \gamma} \left(\frac{y}{y_b} \right)^\gamma \text{ and } \xi = g^*(y_\xi^*) = \frac{\bar{\pi} c}{\Lambda_J \rho \gamma} \left(\frac{y_\xi^*}{y_b} \right)^\gamma.$$

Then (41) and (42) are equivalent to

$$\frac{\frac{d}{dt} F_t}{F_t} = \frac{1}{\gamma} \frac{\frac{d}{dt} F_t^\gamma}{F_t^\gamma} = \frac{1}{\gamma} \frac{\gamma \xi \left(f_\xi^* Y_t \right)^\gamma - \gamma \xi F_t^\gamma}{F_t^\gamma} = \xi \left(\frac{y_t}{y_\xi^*} \right)^\gamma - \xi = \frac{\bar{\pi} c}{\Lambda_J \rho \gamma} \left(\frac{y_t}{y_b} \right)^\gamma - \xi = g^*(y_t) - \xi.$$

This evolution of debt matches (20) with $G_t = g^*(y_t) F_t$. □

Equation (41) implies that the firm's debt today is a type of "discounted moving average" of the firm's initial debt and a target multiple f_ξ^* of the firm's intervening cash flows. This point becomes transparent in the special case of an initially unlevered firm. Setting $F_0 = 0$ in (41), we have

$$F_t = f_\xi^* \left(\gamma \xi \int_0^t e^{\gamma \xi (s-t)} Y_s^\gamma ds \right)^{1/\gamma}. \tag{43}$$

Because $\gamma \xi \int_0^t e^{\gamma \xi (s-t)} ds = 1 - e^{-\gamma \xi t} \approx \gamma \xi t$ for small t, debt grows gradually with order $t^{1/\gamma}$ for an initially unlevered firm, while the firm's long-run debt level depends on the weighted average of its historical earnings.[40]

In the equilibrium debt dynamics, the weight put on recent cash flows relative to more distant ones is an increasing function of the product $\gamma \xi$. Equivalently, as (42) makes clear, the product $\gamma \xi$ determines the speed of adjustment toward the target level. Intuitively, shorter debt maturity (higher ξ) implies faster repayment of debt principal, allowing leverage to shrink more quickly in the face of declining cash flows. From (35), γ increases with shorter

[40] This continuous debt increment at $F_0 = 0$ is consistent with the infinite growth rate of $g^*(y)$ as $y \to \infty$.

maturity, higher growth, or lower volatility, making the firm more aggressive in adding leverage in response to positive cash flows news. Finally, note from (39) that the only impact of the tax rate $\bar{\pi}$ or the default regime (via $\Lambda_J \geq 1$) is on the target debt-to-income level f^*_ξ.[41]

Proposition 9 demonstrates that once the firm is free to adjust leverage over time, equilibrium debt dynamics depart strongly from the predictions of the standard dynamic trade-off theory literature. In particular, our result that debt slowly but continuously adjusts toward a target leverage level differs from the models in which debt levels are fixed initially (say, Leland (1994, 1998)), or jump periodically due to fixed adjustment costs (say, Goldstein, Ju, and Leland (2001)). Figure 3 simulates the evolution of debt of different maturities for an initially unlevered firm. In each case in this 10-year sample path, the shocks to earnings, and therefore the unlevered value of the firm, are the same.

The top panel of Figure 3 shows that the initial impact of these alternative debt maturities on TEV is slight, with differences only emerging later when leverage becomes high. The slow adjustment of total debt in the middle panel highlights the long persistence and hysteresis in debt levels. This speed of adjustment declines with the debt maturity. Finally, the bottom panel makes clear that the primary driver of market leverage in the short term is fluctuation in the stock price.

These features resemble the evolution of debt most commonly observed in practice.[42] Our model therefore provides a theoretical foundation for partial adjustment models (Jalilvand and Harris (1984), Leary and Roberts (2005), etc.) that are widely used in the empirical capital structure literature. Gradual "under-adjustment" also leads to the well-documented negative relation between leverage and profitability, even for frequent issuers of debt (see Frank and Goyal (2015) and Eckbo and Kisser (2018)).

Consider, for example, the case of five-year debt. While the target level of market leverage is 42%, the firm does not issue this amount immediately. Debt increases quickly at first, but is soon outpaced by increases in firm value. When firm value declines after year 4, leverage overshoots the target, as the firm can reduce leverage no faster than its debt matures. In contrast, with two-year debt, the firm adjusts quickly to a target leverage ratio of 53%, and increases leverage significantly as firm value grows, but when firm value declines sharply in year 5, the firm is not willing to reduce leverage quickly enough and

[41] Indeed, the target debt level decreases with Λ_J, as the increase in the value of shareholders' default option comes at creditors' expense, lowering the debt price. The target debt ratio f^*_ξ increases with the tax rate if the debt maturity is long ($\xi \approx 0$). But if the debt maturity is short and tax rates are high, a tax increase may raise the default boundary y_b sufficiently to cause the target debt-to-income ratio to fall.

[42] For example, Welch (2004) reports that market leverage changes primarily due to stock price fluctuations, and Baker and Wurgler (2002) document that the firm's current leverage depends on its equity price over the past decade or more. Graham and Leary (2011) survey a number of studies that suggest an annual speed of adjustment towards a target leverage ratio of between 10% and 40%. Frank and Shen (2019) report that allowing for heterogeneity in the determination of leverage targets leads to much faster estimates of the mean-reverting adjustment speed.

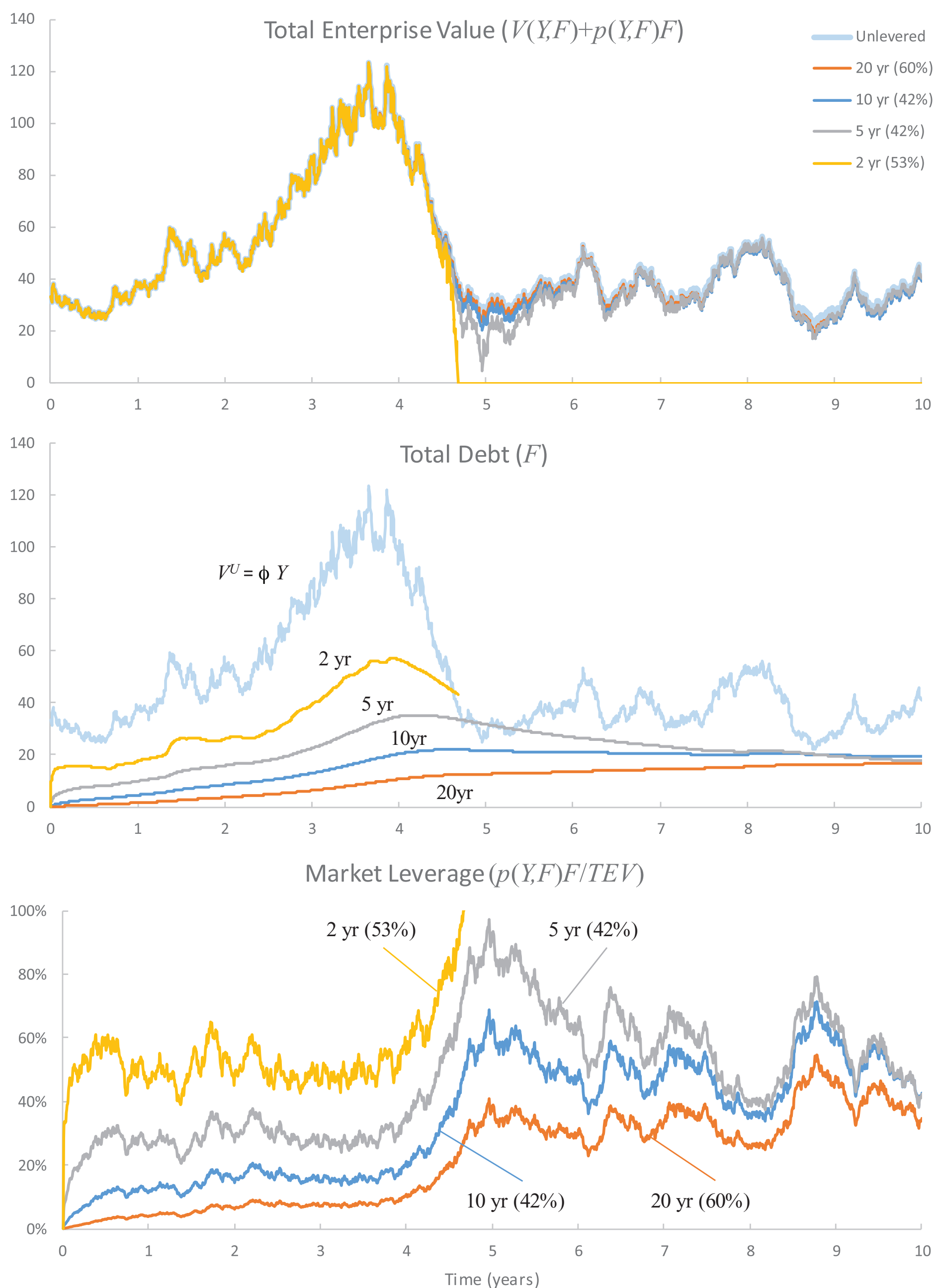

Figure 3. Simulation of debt evolution. This figure shows the evolution of firm value and the debt level for different debt maturities. The top panel plots TEV. The middle panel plots unlevered firm value $V^U = \phi Y$ together with debt levels for different debt maturities. The bottom panel plots market leverage (with the legend indicating the target level). Note that two-year debt defaults before the end of year 5. Parameters: $\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, $c(1 - \bar{\pi}) = r = 5\%$, and $\xi = 5\%, 10\%, 20\%,$ or 50%.

the firm defaults. Finally, with 20-year debt, although the target leverage ratio is 60%, the speed of adjustment is very slow and the firm increases debt gradually over the entire 10-year period.

B. Debt Maturity Indifference

Our model considers a constant maturity structure in which all debt has an expected maturity of $1/\xi$. This assumption is common in much of the dynamic capital structure literature that treats the debt maturity structure as a parameter.[43] While allowing the firm full flexibility over maturity structures is beyond the scope of this paper, we show that absent commitment, shareholders are indifferent to the maturity structure of the firm's future debt issuance. While different maturity choices will lead to different future leverage levels, any increase in tax benefits is offset by an increase in default costs, and the firm's current share price is unaffected.

Recall from Proposition 2 that we can compute the current value of equity as though the firm will not issue or repurchase debt in the future and will just repay its existing debt as it matures. This result immediately implies that for an initially unlevered firm ($F_0 = 0$), firm value does not depend on the choice of debt maturity structure ξ. This irrelevance result can be generalized further. Consider a thought experiment in which shareholders—facing current cash flows and debt structure (Y_t, F_t, ξ)—have a one-time opportunity to choose an alternative maturity ξ' for the firm's *future* debt. That is, the firm's existing debt continues to retire at the old speed ξ, but the newly issued debt retires at the new speed ξ'. We have the following proposition.

PROPOSITION 10: (Maturity Indifference)*: In a no-commitment equilibrium with smooth debt issuance, the firm's current equity value is independent of the maturity ξ' of new debt.*

PROOF: See the Appendix. □

The intuition for the proof is straightforward. We can consider the implied future liabilities from the firm's existing debt as a modification of the cash flow process for the firm and then apply our general methodology as in Section I. For equilibria with smooth debt issuance polices, equity holders obtain zero profit by issuing future debt, and so the equity value would be the same as if the firm could not issue future debt. As a result, the current equity value depends on the maturity structure ξ of existing debt, but not on the maturity structure ξ' of future debt. This logic and hence the indifference result can be further generalized to a setting in which the firm is free to choose any maturity structure for its newly issued debt any time. Again, the equity value will depend only on the maturity structure of the firm's existing debt.

[43] Debt retirement in this fashion is similar to a sinking fund that continuously buys back debt at par. See, for instance, Leland (1998), Leland and Toft (1996), He and Xiong (2012), and Diamond and He (2014). However, see Brunnermeir and Oehmke (2013) and He and Milbradt (2016) for analysis of the firm's decision to lengthen or shorten its debt maturity.

The indifference result can also be seen in Figure 3. Although the debt maturity choice leads to large differences in the evolution of debt and market leverage over time, the initial enterprise value of the firm is the same, and equal to the unlevered firm value in all four cases.

Figure 3 also provides a potential explanation for the finding in Lemmon, Roberts, and Zender (2008) that much of the cross-sectional variation in firms' capital structure is persistent and largely unexplained by observable characteristics. From the perspective of our model, small perturbations or frictions that may lead firms to pick different initial maturity structures will lead over time to dramatically different leverage outcomes. See DeMarzo (2019) for further discussion of this point together with the potential role of collateral.

C. Optimal Maturity: Price Impact and Welfare

In the previous section, we demonstrate that the firm's shareholders are indifferent to the choice of debt maturity when debt issuance is unconstrained. This indifference result regarding debt maturity runs counter to the standard intuition that shareholder-creditor conflicts are ameliorated with short-term debt. While this intuition would hold if shareholders could commit ex ante to maintain a given leverage policy,[44] our analysis shows that without commitment this intuition is not correct: the use of short-term debt induces the firm to lever more aggressively, and the agency costs resulting from the leverage ratchet effect do not disappear. Indeed, as maturity approaches zero ($\xi \to \infty$), the firm's target leverage y_ξ^* converges to the default boundary y_b, increasing the firm's tax shields while solvent but keeping the firm ever closer to default. Intuitively, as long as opportunities to trade are sufficiently frequent relative to the maturity of the debt, leverage ratchet dynamics will reduce the gains from trade.[45]

In this section, we show that a benefit of short-term debt reemerges if the firm is constrained to raise a fixed amount of initial debt. However, while short-term debt may be privately optimal, it may be inefficient from the perspective of social welfare.

C.1. Constrained Borrowing

We have assumed throughout that the firm has frictionless access to both debt and equity markets, and we show that debt issuance will occur gradually over time. Suppose instead the firm is forced to raise a fixed amount of capital

[44] See Tserlukevich (2008) for further elaboration of this point with commitment. The flexibility offered by short-term debt is also studied in a recent paper by Geelen (2017).

[45] Given any fixed maturity $1/\xi > 0$ with continuous trading, there is always the opportunity to issue new debt before the existing debt matures. The same issues arise even in a discrete-time model, as long as new debt can be issued on the same date as the original debt. Bizer and DeMarzo (1992) demonstrate the agency cost associated with sequential rounds of simultaneous borrowing even in a one-period model.

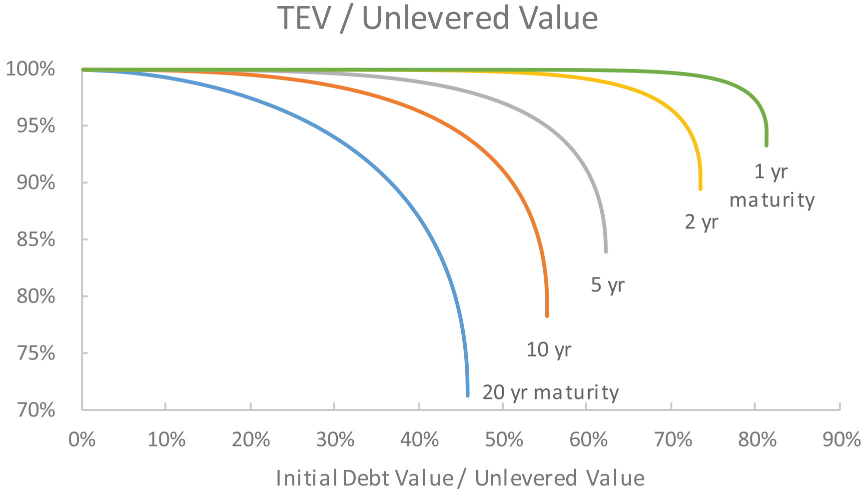

Figure 4. Debt and firm value with differing maturities. If the firm must borrow a fixed amount, using short-term debt raises its debt capacity and reduces the cost to firm value. Parameters: $\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, and $c(1 - \bar{\pi}) = r = 5\%$.

using debt.[46] In that case, which choice of debt maturity would shareholders prefer?

As we have shown earlier, the convexity of the equity value function in F implies that issuing a large block of debt is costly to shareholders. This cost arises because the firm faces downward-sloping demand (i.e., $p_F < 0$) for its debt. The price sensitivity of the debt, however, increases with the debt's maturity because the initial debt issuance will not be actively unwound (due to the leverage ratchet effect), so long-maturity debt with a slower passive retirement speed will be more subject to future dilution. We confirm this result in Figure 4, which shows the drop in total firm value (TEV) for a given initial amount borrowed as a fraction d of the firm's unlevered value,

$$d = \frac{p(Y_0/F)F}{\phi Y_0} = \frac{p(y_0)}{\phi y_0}.$$

Not only does short-term debt reduce the cost to firm value, but it also increases the firm's debt capacity—the maximum amount it is able to borrow—which is restricted because at a high level of debt, the negative price impact from raising the face value more than offsets the increase in quantity.[47]

[46] For example, suppose the firm requires capital to launch and is restricted from using all-equity financing due to governance concerns, illiquidity, or other temporary costs outside the model.

[47] See the NBER working paper version https://www.nber.org/papers/w22799 where we formally establish this result and show that the debt capacity approaches 100% as $\xi \to \infty$. Of course, we ignore other potential costs associated with short-term debt, such as rollover risk (see, for example, He and Xiong (2012)).

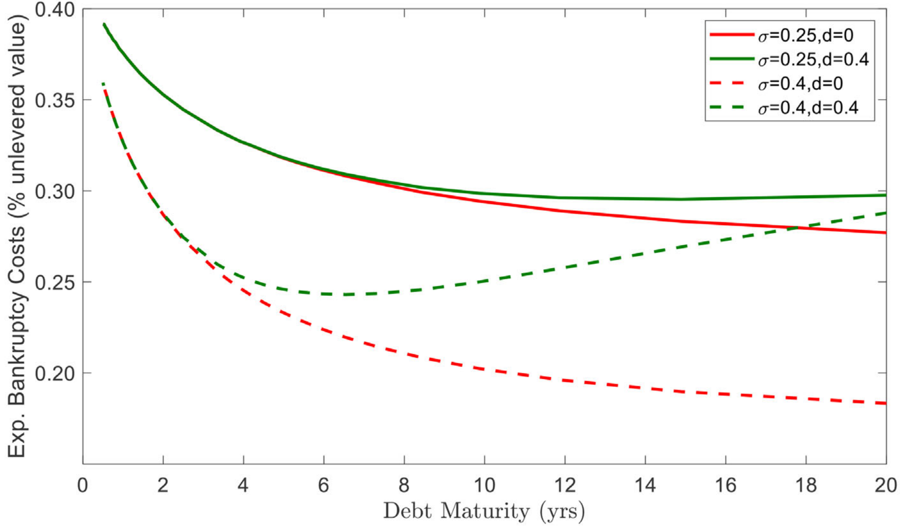

Figure 5. Expected bankruptcy costs *BC* (as a fraction of unlevered firm value). The figure shows that for an initially unlevered firm ($d = 0$), expected bankruptcy costs decrease with both maturity and volatility, due to a slower issuance rate. But if the firm must borrow 40% of its unlevered value upfront ($d = p(y)/(\phi y) = 0.4$), then welfare is maximized for an intermediate debt maturity. Parameters: $r = c(1 - \bar{\pi}) = 5\%$, $\mu = 2\%$, and $\bar{\pi} = 30\%$.

C.2. Maturity and Welfare

The result that shareholders are indifferent to debt maturity given smooth issuance must imply that the expected tax benefits of new debt are exactly offset by the increase in default costs borne by shareholders. But from a social planner's perspective, tax shields represent transfers, while bankruptcy costs reflect dead-weight losses that directly hurt welfare. Therefore, even if shareholders are indifferent to maturity, there will be a clear ranking in terms of expected welfare: Choices that lead to higher expected tax benefits also imply higher expected default costs and hence lower welfare.

We can use our model to calculate the expected future bankruptcy costs as a fraction of the current unlevered firm value, which we denote by $BC(y)$. Starting with initial debt F_0, debt evolves according to $dF_t = (g(y_t) - \xi)F_t dt$ and default occurs at τ_b with $Y_{\tau_b} = y_b F_{\tau_b}$, with a loss equal to the firm's unlevered value at that time. Thus,

$$BC(y) = \frac{1}{\phi Y_0} E\left[\left(e^{-r\tau_b}\phi Y_{\tau_b}\right| Y_0 = yF_0, F_0\right] = \frac{y_b}{y} E\left[\exp\left(\int_0^{\tau_b} (g(y_t) - \xi - r)\, dt\right)\middle| y_0 = y\right].$$

We compute $BC(y)$ for alternative debt maturities in Figure 5.[48] For an initially unlevered firm, expected bankruptcy costs are higher if the debt

[48] We compute BC numerically by defining $H(y) \equiv y \cdot BC(y)$, and noting that H satisfies the ODE $(r + \xi - g(y))H(y) = (\mu + \xi - g(y))yH'(y) + 0.5\sigma^2 y^2 H''(y)$,with boundary conditions $BC(y_b) = 1$ and $BC(y) \to k$, when $y \to \infty$ for some constant $k \in (0, 1)$.

maturity is shorter (i.e., both red curves decline with debt maturity). The intuition for this result is that, as we see in Figure 3, with short-term debt the firm takes on debt more quickly and maintains higher leverage over time, taking greater advantage of interest tax shields but also leading to a higher risk of default. Interestingly, as the figure also shows, lower firm volatility also gives rise to greater expected bankruptcy costs (i.e., the dashed lines sit below the solid lines). The key to this counterintuitive result is the endogenous debt issuance policy. Shareholders in a firm with lower volatility are more aggressive in levering up, so much so that we have greater expected bankruptcy costs.

Figure 5 also reveals that if the firm is constrained to borrow an initial fraction $d \in [0, 1]$ of its unlevered value, then an intermediate debt maturity will maximize welfare (i.e., the green curves are minimized at an interior point). This is because in the case of constrained initial borrowing, Figure 4 shows that given an initial borrowing requirement, longer term debt leads to a larger initial welfare loss *even for shareholders* (i.e., expected bankruptcy costs exceed tax benefits) and shareholders prefer to issue short-term debt. The social planner, however, only cares about bankruptcy costs. Intermediate-maturity debt minimizes default costs by balancing the tradeoff of starting the firm further away from default than longer term debt (given a fixed initial amount raised, as shown in Figure 4), while at the same time reducing the aggressiveness with which it will issue debt in the future compared to short-term debt.

D. Low-Leverage Puzzles

Two important empirical observations associated with leverage are the credit-spread puzzle and the zero-leverage puzzle. As we discuss below, the implications of our model for debt dynamics and pricing can help resolve both of these apparent anomalies.

D.1. The Credit-Spread Puzzle

As Huang and Huang (2012) and others observe, firms with low leverage often have much higher credit spreads than would be predicted by standard structural models. In the context of our model, large credit spreads arise even for firms with low leverage because the debt is priced in anticipation of future debt issuance.

To see this, define the credit spread $\delta(y)$ as the yield spread required to match the bond's price absent default,

$$\frac{c+\xi}{r+\delta(y)+\xi} \equiv p(y), \text{ or equivalently, } \delta(y) \equiv \frac{c+\xi(1-p(y))}{p(y)} - r. \tag{44}$$

In comparison, we can also define the credit spread $\delta^0(y)$ that would apply in a model in which the firm commits not to issue more debt by replacing $p(y)$ with $p^0(y)$ from (30) in definition (44).

It is easy to see that as leverage falls (y increases), both credit spreads decline. But as we approach zero leverage ($y \to \infty$), the credit spread with commitment vanishes ($\delta^0(y) \to 0$), whereas in the no-commitment case,

$$\lim_{y \to \infty} \delta(y) = \frac{\bar{\pi} c}{\rho} > 0. \tag{45}$$

In other words, even with very low leverage, we should expect significant credit spreads in our model. The reason, of course, is that even when the firm's current debt level is very low, the future debt level is likely to be much higher given the mean-reverting leverage dynamics.

D.2. The Zero-Leverage Puzzle

Another significant empirical puzzle for the standard trade-off theory is that despite the tax benefits of debt, there exist a larger number of firms with zero leverage, as documented by Strebulaev and Yang (2013).

At first glance, this fact would also appear to contradict our model, in which firms issue debt repeatedly. However, our equilibrium dynamics and the uniqueness result of Section II only apply in the range $y = Y/F < \infty$, and therefore $F > 0$. When $F = 0$, there are actually two possible MPE outcomes—the firm can begin issuing debt, as in our model, or the firm can remain with zero leverage. In either case, the payoff to shareholders is the same: They receive the unlevered value of the firm.

The zero-leverage equilibrium can exist because without commitment the firm is unable to capture the tax shield benefits of debt. As equation (45) shows, the initial credit spread on the debt is sufficient to offset the tax benefits. But, this no-trade equilibrium disappears once debt is in place. The reason is that, for the price of the firm's outstanding debt to be consistent with equilibrium, the firm must be expected to continue issuing debt. This equilibrium constraint does not apply when $F = 0$, as there is no outstanding debt price.

IV. Endogenous Investment and Debt Overhang

We now extend our model by adding an endogenous investment decision also under the control of shareholders. Including investment allows us to explore the interaction between shareholder-creditor conflicts over investment and leverage choices.

We adjust our model so that investment is efficient and leads to the same cash flow dynamics as in the baseline setting. But if shareholders have the option to cut investment, debt overhang will induce them to do so when leverage is high, as in Myers (1977). Relative to the case without investment options, anticipated agency costs lower the debt price and reduce the rate of debt issuance when leverage is low. But when leverage is high, the option to cut investment delays default and ultimately raises the debt price and rate of issuance. In the end, the tax benefits of debt are offset by the combined costs of

underinvestment and default, so that the share price is again equal to the no-trade value. Hence, because agency costs effectively substitute for default costs, in our model the ex ante value of equity is *not* reduced by the introduction of debt overhang, in stark contrast to standard trade-off models. Rather, our results imply that tax subsidies intended to lower the cost of debt and thereby boost investment may instead have the opposite result.

While these insights are general, for ease of illustration we consider a simple linear investment decision.[49] Specifically, given an investment decision $i_t \in [\mu_0, \mu]$, the firm's EBIT, Y_t, evolves according to

$$dY_t = i_t Y_t dt + \sigma Y_t dZ_t.$$

Hence, when $i_t = \mu$, the firm invests optimally and grows at rate μ as before. But shareholders now have the option to cut investment and choose $i_t < \mu$. If they cut investment, the firm's free cash flow between $(t, t + dt)$ increases, and is given by

$$(1 - \bar{\pi})Y_t dt + \kappa(\mu - i_t)Y_t dt,$$

where parameter $\kappa > 0$ reflects the after-tax cost of investment (i.e., investing an extra \$$\kappa$ today generates a permanent increase to Y_t of \$1). We assume that the investment cost $\kappa < \phi = \frac{1-\bar{\pi}}{r-\mu}$, so an unlevered firm always earns a positive net present value from investing. (The decision not to invest can also be interpreted as a decision to liquidate some existing capital and sacrifice growth.)

Our model with investment thus matches the setting in Section II except for the fact that shareholders now have the option to cut investment. We consider an MPE with the same state variable $y_t = Y_t/F_t$. As before, shareholders default when y drops to an endogenous default threshold y_b^{inv}. In addition, there exists an endogenous investment threshold y_i^{inv} such that if $y \geq y_i^{inv}$, leverage is sufficiently low that shareholders invest efficiently (and cash flows evolve as in the baseline case). The new feature is that now, in the "distressed" region $y \in (y_b^{inv}, y_i^{inv})$, debt overhang is severe enough that shareholders do not find it worthwhile to invest but they are not yet willing to default.

We solve this extension in closed form in Appendix V.D, where we show that the no-trade value remains strictly convex and thus can be used to characterize the smooth equilibrium. We demonstrate the interaction between investment and debt issuance in Figure 6 by comparing the case with endogenous investment (black dotted line) and the baseline setting in which investment is always fixed at first best (blue solid line). We plot the equity value v^{inv}, debt price p^{inv}, enterprise value $v^{inv} + p^{inv}$, and debt issuance rate in Panels A to D, respectively. The dashed vertical lines show the default boundaries for the two settings, whereas the solid vertical line indicates the investment boundary when investment is endogenous.

[49] See the NBER working paper version https://www.nber.org/papers/w22799 for a model with continuous investment and convex adjustment costs.

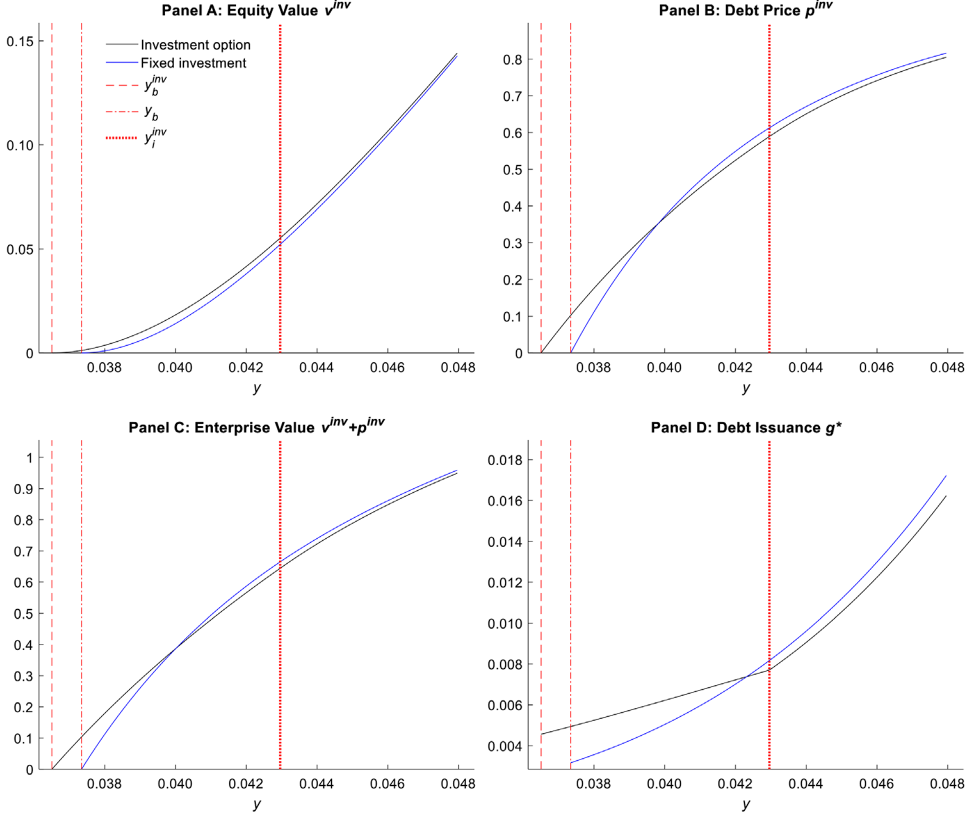

Figure 6. Endogenous investment and debt issuance policies. This figure plots the comparison between the extension with investment options and the baseline model: equity value v (Panel A), debt price p (Panel B), enterprise value $v+p$ (Panel C), and debt issuance polices g^* (Panel D). In the extension with investment options, the firm invests when $y \geq y_i^{inv}$, does not invest when $y \in (y_b^{inv}, y_i^{inv})$, and defaults when y hits y_b^{inv}. In the baseline model, the firm defaults earlier. That is, $y_b > y_b^{inv}$. Parameters: $r = c(1-\bar{\pi}) = 5\%$, $\sigma = 10\%$, $\mu = 2\%$, $\xi = 0.01$, $\kappa = 22$, and $\mu_0 = 0$.

Compared to the baseline case, the option to cut investment and raise the firm's current free cash flow increases the value of equity and lowers the default boundary ($v^{inv} > v$ and $y_b^{inv} < y_b$) as shown in Panel A. In contrast, the option to cut investment lowers the debt price and enterprise value when leverage is low, as creditors anticipate the cost of future underinvestment, but raises them when leverage is high, by delaying default, as shown in Panels B and C.

We plot debt issuance policies in both cases in Panel D. In the investment region $y \geq y_i^{inv}$, shareholders invest in both cases, but issue debt more slowly when there is an option to cut investment in the future. This slower debt issuance is a consequence of the greater sensitivity of the debt price to leverage when creditors anticipate that debt overhang will distort future investment. In the "distressed" region $y \in (y_b^{inv}, y_i^{inv})$, because shareholders can cut investment to boost the firm's current cash flow, the issuance rate declines more slowly as

y falls. Ultimately, close to default, the debt issuance speed without investment exceeds the baseline case. We summarize the key results below.

PROPOSITION 11: (Underinvestment)*: While the unlevered equity value is unchanged, for a given y the option to inefficiently cut investment (or liquidate assets) results in a higher equity value than in the case with fixed investment. Shareholders cut investment when y falls below $y_i^{inv} > y_b$, and default is delayed to $y_b^{inv} < y_b$. When $y \geq y_i^{inv}$, the firm invests optimally, and the value of equity satisfies*

$$v^I(y) = \phi y - \rho - \Lambda_I \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho), \tag{46}$$

with $\Lambda_I > 1$. In this region, the rate of debt issuance is proportionally lower, and the credit spread is higher, than in the fixed investment case.

PROOF: See the Appendix. The intuition is as follows. First, we establish that for an arbitrary investment technology, the no-trade value function remains convex, so the characterization of the smooth equilibrium in Section I can be applied. Given a linear investment cost, the investment decision is "bang-bang" so $i_t \in \{\mu_0, \mu\}$. When y_t is large, $v^{I'}(y) \approx \phi > \kappa$ and investment is efficient ($i_t = \mu$). When y_t falls, shareholders cut investment ($i_t = \mu_0$) whenever debt overhang becomes severe enough that $v^{I'}(y_t) \leq \kappa$, which occurs at the threshold defined by $v^{I'}(y_i^{inv}) \equiv \kappa$. Recall that with zero recovery, $v'(y_b) = 0$, so the option to cut investment is strictly profitable prior to default, and hence $y_b^{inv} < y_b$. We can solve in closed form for the value function v^I in the investment region as in Section II, with the optimal boundary y_i^{inv} determined by the usual smooth-pasting condition. The decision to cut investment is similar to a restructuring decision (as in Section II.D), with the parameters (α, β) determined endogenously from the value function v^{NI} in the no-investment region. Because the value of equity in such a "restructuring" is positive, it will be exercised earlier than when shareholders only have the option to default and receive zero, and thus $y_i^{inv} > y_b$. In the Appendix, we show that, as with our analysis in Section II.D, this disinvestment/restructuring option also implies $\Lambda_I > 1$. □

In conclusion, we show that when shareholders cannot commit to a leverage policy, additional shareholder-creditor conflicts over investment policy have no incremental impact on the firm's initial unlevered value. Instead, the anticipation of investment distortions widens initial credit spreads and slows the initial rate of debt issuance. Ultimately, the cost of the resulting investment distortions substitutes for default costs in fully offsetting the funding cost advantages of debt.

V. Conclusions

When the firm cannot commit ex ante to future leverage choices, shareholders adjust the level of debt to maximize the firm's current share price. We develop a general methodology to solve for equilibrium debt dynamics in this setting, including endogenous investment. When earnings evolve as

geometric Brownian motion (including possible upward jumps), we show the uniqueness of our MPE and explicitly solve for the firm's debt as a slowly adjusting weighted average of past earnings. The endogenous rate of debt issuance decreases with debt maturity and volatility. The firm slows its rate of debt issuance as it approaches default, and thus the firm's equilibrium leverage is ultimately mean-reverting.

Because creditors expect the firm to issue new debt in the future, credit spreads are wider in our model than in standard models with fixed debt and remain wide even when leverage is close to zero. Lower debt prices offset the tax shield benefits of leverage, so that the equity value is identical to the case without no future debt issuance. This inability to capture tax benefits of leverage may provide a possible resolution for the zero-leverage puzzle (Strebulaev and Yang (2013)), as the potential tax benefit from leverage is offset by the high credit spread even for initial debt.

Finally, although shortening the maturity of debt raises both the average level of leverage and its speed of adjustment, the increase in expected tax benefits is again offset by an increase in investment distortions and default costs. As a result, even "instantaneous" debt does not resolve the commitment problem, and shareholders have no incentive to adjust the firm's debt maturity structure. Moreover, because debt adjusts gradually, similar firms may have very different leverage given their exposure to past shocks. These observations offer a potential explanation for findings such as those by Lemmon, Roberts, and Zender (2008) that much of the cross-sectional variation in firms' capital structures is persistent and largely unexplained by firm characteristics.

Many further extensions of our model are worth considering. DeMarzo (2019) allows for different funding costs across equity and debt markets, perhaps due to different investor-level taxes, or a "moneyness" premium associated with debt. DeMarzo, He, and Tourre (2021) consider risk-averse creditors in the context of sovereign debt. As long as debt has a net funding advantage, all of the key results in our model continue to apply. The same holds if we allow for proportional transaction costs associated with debt issuance (which can be interpreted as an additional wedge in the cost of capital across markets).

In our analysis we focus on MPE in which debt pricing depends only on firm fundamentals. As in the folk theorem literature, if we relax this constraint, additional equilibria can be supported using "grim trigger" punishments in response to a deviation. Indeed, because our equilibrium produces the lowest possible equilibrium payoff for shareholders (Markov or not), *all* non-MPE equilibria can be supported by using our equilibrium off the equilibrium path. (See, for example, Benzoni et al. (2020) and Malenko and Tsoy (2020), who use our results to support non-Markov strategies in which the firm is "punished" for exceeding a target leverage ratio by reverting to our MPE.)

Naturally, we expect that firms will try to reduce the agency costs resulting from the leverage ratchet effect and capture some of the funding advantages of debt by using alternative commitment mechanisms such as collateral or covenants that restrict future debt issuance. Our results here imply that commitments to a leverage policy are more important to enhancing firm value than

commitments regarding investment policy. DeMarzo (2019) discusses alternatives and demonstrates that collateral allows the firm to capture the funding advantages of debt because it can be exclusively promised to a single creditor. Other important commitment mechanisms in practice include regulations, restrictions on the tax deductibility of leverage by corporations, and trading frictions (e.g., Benzoni et al. (2020) argue that fixed costs of debt issuance can act as a commitment mechanism allowing the firm to capture tax benefits of debt). Finally, equity market imperfections may prompt the firm to actively manage its internal liquidity (cash) position (as in Hennessy and Whited (2005), Bolton, Chen, and Wang (2011), and Bolton, Wang, Yang (2020)). We leave for future work an exploration of the leverage dynamics that arise from the interaction of these additional forces with the leverage ratchet effects explored here.

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Appendix

A. Remaining Proofs from the Main Text

Proof of Proposition 7: Note that the HJB equation (34) has the linear solution

$$\bar{v}(y) = \phi y - \rho. \tag{A1}$$

The homogeneous delayed differential equation

$$(r+\xi+\lambda)h(y) = (\hat{\mu}+\xi)yh'(y) + \frac{1}{2}\sigma^2 y^2 h''(y) + \lambda h(\theta y)$$

has solutions of the form $y^{-\hat{\gamma}}$, where $\hat{\gamma}$ solves the characteristic equation (35). In (35), because W is convex, $W(\infty) = W(-\infty) = \infty$, $W(0) < -r-\xi < 0$, and $W(-1) = \hat{\mu} - r < 0$, where W has a unique positive real root (as well as a unique negative real root $\hat{\eta} < -1$ that can be ruled out by the upper boundary condition). The remainder of the analysis follows exactly as in Section II.A. (See also Section II.B for an alternative proof in terms of $f = F/Y$.) For the comparative statics, note by convexity and the fact that γ is the largest root that $W'(\gamma) > 0$. Hence, for a given parameter x, sign$\frac{\partial}{\partial x}\gamma = -$sign$\frac{\partial}{\partial x}W(\gamma)$, where

$$W(\gamma) = \lambda\theta^{-\gamma} + \frac{1}{2}\sigma^2\gamma^2 - (\mu - \lambda(\theta-1) + \xi - \frac{1}{2}\sigma^2)\gamma - (r+\xi+\lambda) = 0.$$

So, for example, because $\theta > 1$ and $\gamma > 0$,

$$\frac{\partial}{\partial\theta}W(\gamma) = -\gamma\lambda\theta^{-(1+\gamma)} + \lambda\gamma > 0,$$

and thus we have that γ is strictly decreasing in θ. □

Proof of Proposition 8: As in the proof of Proposition 4, the equity value function given no trade is given by

$$
\begin{aligned}
v^J(y) &= \max_{\hat{y}_b} \ \phi y - \rho + \left(\frac{y}{\hat{y}_b}\right)^{-\gamma} \left(v^B(\hat{y}_b) - (\phi \hat{y}_b - \rho)\right) \\
&= \max_{j,\hat{y}_b} \ \phi y - \rho + \left(\frac{y}{\hat{y}_b}\right)^{-\gamma} \left((\alpha_j \phi \hat{y}_b - \beta_j \rho) - (\phi \hat{y}_b - \rho)\right) \\
&= \ \phi y - \rho - y^{-\gamma} \left[\min_{j,\hat{y}_b} \hat{y}_b^{\gamma} \left((1-\alpha_j)\phi \hat{y}_b - (1-\beta_j)\rho\right)\right].
\end{aligned}
$$

From the first-order condition for $\hat{y}_b$,

$$
\hat{y}_b = \frac{(1-\beta_j)\rho\gamma}{(1-\alpha_j)\phi(1+\gamma)} = \frac{1-\beta_j}{1-\alpha_j} y_b.
$$

Hence,

$$
\begin{aligned}
v^J(y) &= \ \phi y - \rho - y^{-\gamma} \left[\min_j \hat{y}_b^{\gamma} (1-\beta_j)(\phi y_b - \rho)\right] \\
&= \ \phi y - \rho - \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho) \left[\max_j \left(\frac{1-\beta_j}{1-\alpha_j}\right)^{\gamma} (1-\beta_j)\right] \\
&= \ \phi y - \rho - \Lambda_J \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho).
\end{aligned}
$$

The remaining results follow from Proposition 3 as in Section II.A. □

Proof of Proposition 10: The result follows from a straightforward extension of the methodology in Section I. Let $F = (F_o, F_n)$ be a vector showing the quantity of old and new debt, respectively. Let $c = (c_o, c_n)$ and $\xi = (\xi_o, \xi_n)$ be the corresponding coupon and amortization rates. Given (Y, F), the firm generates cash at rate $u(Y, F) = Y - \pi(Y - c \cdot F) - (c + \xi) \cdot F$. Let $p^n(Y, F)$ be the price of the new debt and $V(Y, F)$ be the value of equity. Then the equity value function satisfies the HJB,

$$
\underbrace{rV}_{\text{required return}} = \max_G \left[u(Y,F) + \underbrace{Gp^n}_{\text{new debt issuance}} + \underbrace{(G - \xi_n F_n) V_{F_n} - \xi_o F_o V_{F_o}}_{\text{evolution of } dF} + \underbrace{\mu(Y) V_Y + \frac{1}{2}\sigma(Y)^2 V_{YY}}_{\text{evolution of } dY} \right].
$$

As before, if a smooth issuance policy is optimal, it must be the case that $p^n = V_{F_n}$. Thus, the HJB becomes

$$
rV = u(Y,F) - \xi_n F_n V_{F_n} - \xi_o F_o V_{F_o} + \mu(Y) V_Y + \frac{1}{2}\sigma(Y)^2 V_{YY},
$$

equivalent to the case with no future trade. Note that before the firm issues new debt, $F_n = 0$ and hence the value of equity $V(Y, (F_o, 0)) = V(Y, F_o)$ is independent of coupon rate or maturity (c_n, ξ_n) of the new debt. Finally, following

the same approach as in Section I, we can solve for the optimal G^* as before,

$$G^* = \frac{\frac{\partial}{\partial F_n} u(Y, F) + c_n + \xi_n}{-p^n_{F_n}} = \frac{\pi' c_n}{-p^n_{F_n}}.$$

Thus, the smooth equilibrium will exist under the same conditions as before (differentiability and strict convexity with respect to the debt face value). □

B. Solving the Model in Terms of f

In some cases it is more convenient to solve the model in terms of the state variable $f = F/Y = 1/y$, especially when considering an initially unlevered firm ($F = 0$). Let $\hat{v}(f)$ be the value function in terms of f, that is, $\hat{v}(f) = V(1, f) = V(Y, F)/Y$. Note that given a jump, the value becomes $V(\theta Y, F)/Y = \theta \hat{v}(f/\theta)$. Hence, the HJB equation for $\hat{v}(f)$ given no trade is

$$(r - \hat{\mu})\hat{v}(f) = (1 - \bar{\pi})\left(1 - cf\right) - \xi f - (\hat{\mu} + \xi) f \hat{v}'(f) + \tfrac{1}{2}\sigma^2 f^2 \hat{v}''(f) + \lambda\left(\theta\hat{v}\left(\frac{f}{\theta}\right) - \hat{v}(f)\right). \tag{A2}$$

Recall that $\mu = \hat{\mu} + \lambda(\theta - 1) < r$. According to Liu (2018), Theorem 2.2, the general solution of (A2) is of the form

$$\hat{v}(f) = \phi - \rho f + A_1 f^{\hat{\gamma}} + A_2 f^{-\hat{\eta}}, \tag{A3}$$

with A_1 and A_2 as coefficients to be determined. The two power functions of $f^{\hat{\gamma}}$ and $f^{-\hat{\eta}}$ are derived as follows. The characteristic equation of the homogeneous part is

$$\hat{W}(x) = \lambda\theta^{1-x} + 0.5\sigma^2 x^2 - \left(\hat{\mu} + \xi + 0.5\sigma^2\right)x - (r - \hat{\mu} + \lambda) = 0.$$

Because $\hat{W}''(x) = \lambda\theta^{1-x}(\ln\theta)^2 + \sigma^2 > 0$, $\hat{W}(x)$ is convex. In addition, because $\hat{W}(-\infty) = \hat{W}(+\infty) = \infty$ and both $\hat{W}(0) = -(r - \hat{\mu}) < 0$ and $\hat{W}(1) = -\xi - r < 0$, we have two real roots $\hat{\gamma} > 1$ and $-\hat{\eta} < 0$ for the equation $\hat{W}(x) = 0$. It is easy to check that $\hat{\gamma} = 1 + \gamma$ from (35).

Because $\hat{v}(0)$ is bounded, we must have $A_2 = 0$ and $\hat{v}(0) = \phi$. Thus, A_1 is determined from the boundary condition $\hat{v}(f_b) = \hat{v}'(f_b) = 0$, which implies

$$\hat{v}(f) = \phi - \rho f - \left(\frac{f}{f_b}\right)^{\hat{\gamma}}\left(\phi - \rho f_b\right) \text{ with } f_b = \frac{\hat{\gamma}}{\hat{\gamma} - 1}\frac{\phi}{\rho}.$$

It is straightforward to check that $f_b = 1/y_b$ and $\hat{v}(f) = v(y)/y$.

C. Equilibrium Uniqueness

We prove that the smooth equilibrium constructed in Section II is the unique equilibrium in the class of MPE with the unidimensional state variable being $y = Y/F$. Our proof strategy comprises five steps.

(1) The equity value $v(y)$ is weakly convex and continuously differentiable C^1, and the debt price function $p(y)$ is continuous and increasing.[50]

These regularity conditions are satisfied by the equity value and debt price in any equilibrium. Most of them are straightforward implications of Proposition 1, but use the strength of having only a single state variable y to rule out discontinuities in the debt price.

(2) There will be no buybacks in any Markov equilibria with Markov state (Y, F).

We show that by deferring any planned buybacks, the firm will benefit from tax shields and its default option. This step is crucial, as it allows us to focus on monotone issuance polices. It also does not depend on the assumption that the equilibrium is Markov in y.

(3) The state variable y_t is a smooth process, plus some singular nonincreasing process.

Following on Step 2, we use the Lebesgue decomposition theorem for, which says that a monotone process can be decomposed into an absolutely continuous part and a singular part. We further show that if a singular part emerges in equilibrium, then there exists a point $\hat{y}$ so that $v''(\hat{y}) = 0$.

(4) There are no gains from trade for an initially unlevered firm.

Here we show that in any equilibrium, $V(Y, 0) = \phi Y$, and thus there are no gains from trade for an initially unlevered firm.

(5) The equilibrium issuance policy is absolutely continuous.

Having shown that the equilibrium payoff for an unlevered firm is unchanged, the last step is to rule out the possibility of singular strategies in equilibrium.

Taken together, these steps verify that our constructed equilibrium—in which the debt issuance policy is absolute continuous in time (i.e., smooth)—is the unique MPE in y.

C.1. Convexity and Differentiability of the Equity Value

We first prove some regularity properties for the scaled equity value function $v(y) = V(Y, F)/F$. Denote by y_b the default policy in any equilibrium, which, with a slight abuse of notation, could differ from the no-trade default boundary y_b^0 (in our constructed equilibrium, they coincide as in equation (26)).

[50] Recall that we use the term "increasing" in the weak sense throughout the paper (see footnote 11).

LEMMA C.1: *We have the following properties:*

(a) There exists a default boundary y_b such that shareholders continue whenever $y > y_b$ with $y_b < y_b^0$;
(b) $v(y)$ is increasing, weakly convex and continuously differentiable C^1, and the debt price function $p(y) = yv'(y) - v(y)$ is continuous and increasing over the region $y \in [y_b, \infty)$; and
(c) When v is linear in some interval, $p(y)$ is a constant and positive in that interval.

PROOF: To start, because $v(y) \geq v^0(y) > 0$ for $y \geq y_b^0$, we know that it is optimal to continue for $y \geq y_b^0$, implying that $y_b < y_b^0$. The optimality of the proposed threshold strategy follows from the monotonicity and convexity established in (b).

For (b), the monotonicity of $v(y)$ comes from $V(Y, F)$ being decreasing in F as shown in Proposition 1. Further,

$$V(Y, F) \geq V(Y, \hat{F}) + (\hat{F} - F)p(Y, \hat{F}) \Rightarrow \frac{v(y)}{y} \geq \frac{v(\hat{y})}{\hat{y}} + \left(\frac{1}{\hat{y}} - \frac{1}{y}\right) p(\hat{y}).$$

Define $q(y) \equiv \frac{v(y)+p(y)}{y} > 0$. Then the inequality above is equivalent to

$$v(y) \geq v(\hat{y}) + (y - \hat{y})\, q(\hat{y}),$$

implying that v is weakly convex in y and q is a subgradient.

Next, the value function must be smooth (continuously differentiable) in Y; otherwise, at a kink there would be an infinite expected rate of gain given the Brownian motion in Y (note that shareholders have the option to issue no debt, and hence reap this infinite gain, if it exists). As a result, v is C^1 in y, which implies that $v' = q$, and hence $p = yv' - v$ follows. Since Proposition 1 shows that $p(Y, F)$ is decreasing in F, it follows that in our MPE, $p(y)$ is increasing in y.

For (c), in any interval (y_1, y_2) with $y_2 > y_1 \geq y_b$, if $v(y) = v(y_1) + v'(y_1)(y - y_1)$ for $y \in (y_1, y_2)$, then in this interval the debt price $p(y) = yv'(y) - v(y)$ must be equal to $yv'(y_1) - v(y_1) - v'(y_1)(y - y_1) = y_1 v'(y_1) - v(y_1)$, which is a constant independent of y. Finally, $p(\cdot)$ is positive because in equilibrium default occurs immediately if and only if $p = 0$. □

C.2. No Buybacks

As mentioned in Proposition 1, the result of no debt buyback generally holds for any Markov equilibrium, with the Markov states being exogenous cash flow Y and endogenous debt face value F. Hence, this constitutes the formal proof for the first result in Proposition 1.

The key idea behind the proof is as follows. Given a potential equilibrium debt issuance policy with buybacks, we show that equity holders can strictly improve their value by following an alternative strategy without buybacks

taking the price function $p(Y, F)$ as given. For example, consider a strategy in which the firm buys back \$1 (face value) of debt at date t and then issues \$3 of debt at date $t+1$. Shareholders would strictly gain by postponing the buy-back at t and issuing only \$2 at $t+1$—the final debt level and debt price (by the Markov assumption) are the same, but the firm earns the incremental tax shield from \$1 of debt between date t and $t+1$. In addition, shareholders retain the option to default on the additional \$1 of debt, further enhancing their payoff.

Formally, suppose there exists an equilibrium starting from $F_0 \geq 0$ with $dF_t = -\xi F_t dt + d\Gamma_t$, or equivalently, $F_t = e^{-\xi t}F_0 + \int_0^t e^{\xi(s-t)}d\Gamma_s$. The associated bond valuation equation implies that the bond price $p_t = p(Y_t, F_t)$ satisfies

$$(r+\xi)\,p_t dt = (c+\xi)\,dt + E_t\left[dp_t\right] \Leftrightarrow E_t\left[rp_t dt - dp_t\right] = cdt + \xi\,(1-p_t)\,dt. \tag{A4}$$

Consider an alternative policy without buybacks defined by

$$\hat{F}_t \equiv \sup_{s\leq t}\{e^{-\xi(t-s)}F_s\} \geq F_t.$$

Because $d\hat{\Gamma}_t = d\hat{F}_t + \xi\hat{F}_t dt$, one can show that

$$d\hat{\Gamma}_t = \max\left(0, F_t - \hat{F}_{t-}\right) = \max\left(0, d\Gamma_t + \int_0^{t-} e^{\xi(s-t)}\left(d\Gamma_s - d\hat{\Gamma}_s\right)\right) \geq 0, \tag{A5}$$

which features no buybacks. Intuitively, this policy postpones any buybacks, issuing debt only as needed to prevent the new debt level $\hat{F}_t$ from ever falling below the original debt level F_t. We will show that if the original policy has debt buybacks, then shareholders would gain by deviating to this new policy, and so the policy with buybacks cannot be an equilibrium.

Shareholders take the debt price function $p(Y, F)$ as given, and so anticipate a debt price of $\hat{p}_t \equiv p(Y_t, \hat{F}_t)$ under the new debt policy. Let $d\Gamma_t^{\Delta} \equiv d\hat{\Gamma}_t - d\Gamma_t$ be the deviation of the issuance policy, and $F_t^{\Delta} \equiv \hat{F}_t - F_t$ be the deviation of the debt path, which equals the cumulative deviation of debt issuances,

$$F_t^{\Delta} = \int_0^t e^{\xi(s-t)}d\hat{\Gamma}_s - \int_0^t e^{\xi(s-t)}d\Gamma_s = \int_0^t e^{\xi(s-t)}d\Gamma_s^{\Delta}. \tag{A6}$$

We then have

$$dF_t^{\Delta} = d\Gamma_t^{\Delta} - \xi F_t^{\Delta}dt \Leftrightarrow d\Gamma_t^{\Delta} = dF_t^{\Delta} + \xi F_t^{\Delta}dt. \tag{A7}$$

Note that if shareholders deviate to $\hat{F}$, they may also reoptimize their default policy accordingly. Doing so can only further improve the payoff from deviating. Hence, to show that the deviation is profitable, it is sufficient to show that the nonnegative issuance policy $d\hat{\Gamma}$ dominates $d\Gamma$, given the default timing τ_b, as a function of cash-flow history $\{Y_t\}$, under the original issuance policy. We establish this below.

LEMMA C.2: *We have*

$$F_t^{\Delta} \geq 0 \text{ for } \forall t \in [0, \tau_b], \text{ with } F_0^{\Delta} = 0, \text{ and} \tag{A8}$$

$$\hat{p}_t d\hat{\Gamma}_t - p_t d\Gamma_t = p_t \left(d\hat{\Gamma}_t - d\Gamma_t \right) = p_t d\Gamma_t^{\Delta}. \tag{A9}$$

PROOF: The claim in (A8) is obvious. Because issuance only occurs under the new policy in order to keep $\hat{F}_t$ from falling below F_t, we have $d\hat{\Gamma}_t > 0 \Rightarrow \hat{F}_t = F_t \Rightarrow \hat{p}_t = p_t$, where the final implication follows from the Markov property that the debt price depends on Y and F only (and not their history). Although the debt price may differ when $d\hat{\Gamma}_t = 0$, the total issuance proceeds are not affected, and hence $\hat{p}_t d\hat{\Gamma}_t = p_t d\hat{\Gamma}_t$, proving (A9). □

We are ready to show the gain from postponing buybacks (fixing the timing of default). Define

$$\pi_t^{\Delta} \equiv \pi(Y_t - c\hat{F}_t) - \pi(Y_t - cF_t) \leq 0,$$

and note that, because π is strictly decreasing and $\hat{F}_t \geq F_t$, π_t^{Δ} is strictly negative whenever $\hat{F}_t \neq F_t$. The change in the value of equity can then be expressed as

$$\begin{aligned}
V_0^{\Delta} = \hat{V}_0 - V_0 &= E\left\{\int_0^{\tau_b} e^{-rt}\left((c+\xi)\left(F_t - \hat{F}_t\right) + \pi(Y_t - cF_t) - \pi(Y_t - c\hat{F}_t)\right)dt + e^{-rt}\left(\hat{p}_t d\hat{\Gamma}_t - p_t d\Gamma_t\right)\right\} \\
&\overset{\text{Eq (A8)}}{=} E\int_0^{\tau_b}\left[e^{-rt} p_t d\Gamma_t^{\Delta} - e^{-rt}\left((c+\xi)F_t^{\Delta} + \pi_t^{\Delta}\right)dt\right] \\
&\overset{\text{Eq (A6)}}{=} E\int_0^{\tau_b}\left[e^{-rt} p_t\left[dF_t^{\Delta} + \xi F_t^{\Delta} dt\right] - e^{-rt}\left((c+\xi)F_t^{\Delta} + \pi_t^{\Delta}\right)dt\right] \\
&= E\int_0^{\tau_b} e^{-rt} p_t dF_t^{\Delta} - E\int_0^{\tau_b} e^{-rt}\left(c + \xi\left(1 - p_t\right)\right)F_t^{\Delta} dt - E\int_0^{\tau_b} e^{-rt}\pi_t^{\Delta} dt.
\end{aligned}$$

Integration by parts for the first term $E\int_0^{\tau_b} e^{-rt} p_t dF_t^{\Delta}$ gives

$$\begin{aligned}
E\left\{\int_0^{\tau_b} e^{-rt} p_t dF_t^{\Delta}\right\} &= E\left\{e^{-r\tau_b} p_{\tau_b} F_{\tau_b}^{\Delta} - p_0 F_0^{\Delta} - \int_0^{\tau_b} F_t^{\Delta}\cdot e^{-rt}\left[E_t\left(dp_t\right) - rp_t dt\right]\right\} \\
&\overset{\text{Eq (49)}}{=} E\left\{e^{-r\tau_b} p_{\tau_b} F_{\tau_b}^{\Delta} - p_0 F_0^{\Delta} + \int_0^{\tau_b} F_t^{\Delta}\cdot e^{-rt}\left(c + \xi\left(1 - p_t\right)\right)dt\right\}.
\end{aligned}$$

Plugging this result back into our calculation of V_0^{Δ}, and using (A8) together with the fact that $p_{\tau_b} = 0$ under the original policy, we have

$$V_0^{\Delta} = E\left\{e^{-r\tau_b}\underbrace{p_{\tau_b}}_{=0} F_{\tau_b}^{\Delta} - p_0\underbrace{F_0^{\Delta}}_{=0} - E\int_0^{\tau_b} e^{-rt}\pi_t^{\Delta} dt\right\} = -E\int_0^{\tau_b} e^{-rt}\pi_t^{\Delta} dt \geq 0,$$

where the inequality is strict unless $\hat{F}_t = F_t$ and the original policy had no buybacks. Otherwise, shareholders strictly gain by capturing the debt tax shield on any deferred buybacks. This completes the proof that there are no buybacks in equilibrium.

C.3. Smooth versus Singular Issuance

The important no-buyback result implies that the equilibrium issuance policy Γ_t must be nondecreasing over time. By Lebesgue's decomposition theorem for monotonic functions (cf. Theorem 19.61, § 19, Chapter V, Hewitt and Stromberg (1965)), we can decompose Γ_t into a smooth and singular process,

$$\Gamma_t = \Gamma_t^{\text{a.c.}} + \underbrace{\Gamma_t^{\text{c.s.}} + \Gamma_t^{\text{jump}}}_{\Gamma_t^{\text{singular}}}. \tag{A10}$$

Here, $\Gamma_t^{\text{a.c.}}$ is an absolutely continuous process, so there exists some positive process G_t such that $d\Gamma_t^{\text{a.c.}} = G_t dt$. The component $\Gamma_t^{\text{c.s.}}$ captures continuous but singular increases that occur at isolated points on the state space (e.g., a "devil's staircase") and $d\Gamma_t^{\text{jump}}$ captures discrete jumps. We aim to rule out both $\Gamma_t^{\text{c.s.}}$ and Γ_t^{jump}, which we refer to as $\Gamma_t^{\text{singular}}$, in any equilibrium, and thereby establish the desired result.

Under our assumption that Y_t evolves according to a geometric Brownian motion and the equilibrium is Markov perfect in y, the decomposition in (A10) implies that we can write

$$dy_t = d\left(\frac{Y_t}{F_t}\right) = y_t\left(\hat{\mu} - \xi + g(y_t)\,dt\right) + y_t\sigma dZ_t + (\theta - 1)y_{t-}dN_t - dL_t^{\text{singular}}$$

for some singular increasing process $L_t^{\text{singular}} = L_t^{\text{c.s.}} + L_t^{\text{jump}} > 0$. Here, $L_t^{\text{c.s.}}$ (L_t^{jump}) occurs if and only if $\Gamma_t^{\text{c.s.}}$(Γ_t^{jump}) occurs.

We now show that the existence of a nonzero singular component to the optimal strategy implies the existence of $\hat{y} > y_b$ such that $v''(\hat{y}) = 0$. There are two cases to consider.

First, consider L_t^{jump}. By Lemma 1 in Section I.B, the optimal issuance strategy can jump only in an interval for which $V(Y, F)$ is linear in F and therefore, by Proposition 1, the debt price is constant. In our homogeneous setting, $y^2 v''(y) = FV_{FF}$, and so a jump can only occur in an interval for which $v(y)$ is linear. Let $\hat{y}$ be the lower bound of this linear segment. Because it is not optimal to jump to default, we can assume $\hat{y} > y_b$. Note that $p'_+(\hat{y}) = 0$, and therefore $p'_-(\hat{y}) = 0$, as otherwise the debt price would have an infinite expected loss rate at $\hat{y}$, violating no-arbitrage. Thus, $p'(\hat{y}) = v''(\hat{y}) = 0$.

Second, consider an isolated singular point $\hat{y}$ for which $L_t^{\text{c.s.}} \geq 0$ (note that $L_t^{\text{c.s.}} > 0 \Leftrightarrow \Gamma_t^{\text{c.s.}} > 0$ and $L_t^{\text{c.s.}} = 0 \Leftrightarrow \Gamma_t^{\text{c.s.}} = 0$; see the discussion after Eq. (A10) which explains the connection between Γ_t and L_t). Based on the study of "skew Brownian motion" in Harrison and Shepp (1981), if $l_{\hat{y}}^{y}(t)$ is the local time that

the process $\{y_t\}$ spends at $y_t = \hat{y}$, then we have for some positive constant $\chi \in (0, 1]$,

$$L_t^{\text{c.s.}} = \chi \cdot l_{\hat{y}}^{y}(t). \tag{A11}$$

The speed of change is proportional to local time (continuous in time but faster than dt, as it is not absolutely continuous in time). As a result, the excursions of the state variable are asymmetric, with probability $\frac{1}{2}(1 + \chi)$ to the left and probability $\frac{1}{2}(1 - \chi)$ to the right. The no-arbitrage condition for bond investors requires that the expected local gain at $\hat{y}$ be zero. Therefore,

$$\frac{1}{2}(1 - \chi)\, p'_{+}(\hat{y}) = \frac{1}{2}(1 + \chi)\, p'_{-}(\hat{y}). \tag{A12}$$

Because the singular point $\hat{y}$ must be isolated, there is an open ball $B \equiv (\hat{y} - \varepsilon, \hat{y} + \varepsilon)$ such that $dL_t^{\text{singular}} = 0$ on $B \backslash \hat{y}$. Because v and v' are continuous, and since the HJB equation for v holds and is identical on the left-hand and right-hand sides of $\hat{y}$, we must have that v'' is continuous at $\hat{y}$, and therefore $p'_{+}(\hat{y}) = p'_{-}(\hat{y}) = v''(\hat{y}) = 0$.

C.4. No Gains from Trade

Next we prove that in any equilibrium, an initially unlevered firm achieves no gains from trade and thus has value ϕY. Specifically,

$$\frac{V(Y, 0)}{Y} = \lim_{F \to 0} \frac{V(Y, F)}{Y} = \lim_{y \to \infty} \frac{v(y)}{y} = \lim_{y \to \infty} v'(y) = \phi. \tag{A13}$$

We must consider three cases regarding the trading strategy for the unlevered firm: (i) trading is smooth in a neighborhood of $F = 0$, (ii) the firm immediately issues debt $F > 0$, or (iii) trading is singular in any neighborhood of $F = 0$.

Case (i): Trading is smooth in a neighborhood of $F = 0$.

In this case, we can use the fact that $V(Y, 0) = \hat{v}(0)Y$ and apply the results of Section II.B. Because trading is smooth, the HJB equation (A2) applies in a neighborhood of $f = 0$ and has solution of the form (A3). Because $V(Y, 0)$ is bounded, $A_2 = 0$ and therefore $\hat{v}(0) = \phi$.

Case (ii): The firm immediately issues debt $F > 0$.

For a jump to be optimal, there exists some largest $\hat{y}$ such that $v(y)$ is linear for $y \geq \hat{y}$. Differentiating the HJB equation (34) and using linearity plus

the fact that $v''(\hat{y}) = 0$ from Step C.3, and therefore $v'''_{-}(\hat{y}) \leq 0$ thanks to the convexity of $v\,(.)$

$$(r+\xi)\,v'\,(\hat{y}) = 1-\bar{\pi} + (\hat{\mu}+\xi)\left[\hat{y}\,\underbrace{v''\,(\hat{y})}_{0} + v'\,(\hat{y})\right] + \frac{1}{2}\sigma^2\left[\hat{y}^2\,\underbrace{v'''_{-}\,(\hat{y})}_{\leq 0} + 2\hat{y}\,\underbrace{v''\,(\hat{y})}_{0}\right]$$
$$+\lambda\left(\underbrace{\theta v'\,(\theta\hat{y})}_{\theta v'(\hat{y})} - v'\,(\hat{y})\right)$$
$$\leq 1-\bar{\pi} + (\hat{\mu}+\xi)\,v'\,(\hat{y}) + \lambda\,(\theta-1)\,v'\,(\hat{y}) = 1-\bar{\pi} + (\mu+\xi)\,v'\,(\hat{y})\,.$$

Therefore,

$$v'\,(\hat{y}) \leq \frac{1-\bar{\pi}}{r-\mu} = \phi. \tag{A14}$$

But because $V(Y,0) \geq \phi Y$, since the firm always has the option not to trade, we must have $v'(y) = \phi$ for $y \geq \hat{y}$.

Case (iii): Trading is singular in any neighborhood of $F = 0$.

In this case, using the results of Step C.3, there exists an increasing sequence of points $\{\hat{y}_n \to \infty\}$ such that $v''(\hat{y}_n) = 0$ and $v'''_{-}(\hat{y}) \leq 0$. Differentiating the HJB equation (34) as in Case (ii), but without assuming linearity in the jump region,

$$(r+\xi)\,v'\,(\hat{y}_i) \leq 1-\bar{\pi} + (\hat{\mu}+\xi)\,v'\,(\hat{y}_i) + \lambda\,(\theta-1)\,v'\,(\hat{y}_i) + \lambda\theta\left(v'\,(\theta\hat{y}_i) - v'\,(\hat{y}_i)\right)$$
$$= 1-\bar{\pi} + (\mu+\xi)\,v'\,(\hat{y}_i) + \lambda\theta\left(v'\,(\theta\hat{y}_i) - v'\,(\hat{y}_i)\right)$$

Taking the limit as $\hat{y}_i \to \infty$, and because $\lim_{y\to\infty}\ v'(y)$ converges monotonically (from convexity and the fact that $V(Y,0)$ is bounded in (A13)), we get

$$\lim_{y\to\infty}\ v'(y) \leq \frac{1-\bar{\pi}}{r-\mu} = \phi.$$

Again, because the firm has the option not to trade, we must have $\lim_{y\to\infty}\ v'(y) = \phi$.

C.5. Smooth Issuance

Now that we have established that there are no gains from trade, we complete the proof that the smooth equilibrium is the unique MPE by showing that the equilibrium trading strategy does not involve singularities.

We begin with a result that bounds the potential gain from a cash flow jump.

LEMMA C.3: (Gains from Cash Flow Jumps): *For $\theta \geq 1$ and any two points y and y_1 such that $y_1 \geq y \geq y_b$, we have*

$$v(y) \geq v(\theta y) - v'(\theta y)(\theta - 1)y \geq v(\theta y) - \phi(\theta - 1)y \tag{A15}$$

$$v(\theta y_1) \geq v(\theta y) + v'(\theta y)(y_1 - y) \geq v(\theta y) + \theta v'(y)(y_1 - y). \tag{A16}$$

PROOF: In both cases, the first inequality follows from convexity, and the second from the fact that v' is monotonically increasing and, from Step C.4, bounded by ϕ. □

Next, suppose there exists a singular component to the optimal trading strategy. From Step C.3, there exists $\hat{y}$ with $v''(\hat{y}) = 0$ and for which trading is smooth (and the HJB holds) in the neighborhood just below $\hat{y}$. Evaluating the HJB of $v(\cdot)$ at $\hat{y}$,

$$\begin{aligned}(r+\xi)v(\hat{y}) &= (1-\bar{\pi})(\hat{y}-c) - \xi + (\hat{\mu}+\xi)\hat{y}v'(\hat{y}) + \frac{1}{2}\sigma^2\hat{y}^2 v''(\hat{y}) + \lambda(v(\theta\hat{y}) - v(\hat{y}))\\ &= (1-\bar{\pi})(\hat{y}-c) - \xi + (\hat{\mu}+\xi)\hat{y}v'(\hat{y}) + \lambda(v(\theta\hat{y}) - v(\hat{y}))\\ &\leq (1-\bar{\pi})(\hat{y}-c) - \xi + (\hat{\mu}+\xi)\hat{y}\phi + (\hat{\mu}+\xi)\hat{y}\left(v'(\hat{y}) - \phi\right) + \lambda(\theta-1)\phi\hat{y}\\ &= \underbrace{(1-\bar{\pi})}_{(r-\mu)\phi}\hat{y} + \left(\underbrace{\hat{\mu}+\lambda(\theta-1)}_{\mu} + \xi\right)\hat{y}\phi - (c(1-\bar{\pi}) + \xi)\\ &\quad + (\hat{\mu}+\xi)\hat{y}\left(v'(\hat{y}) - \phi\right)\\ &= (r+\xi)(\phi\hat{y} - \rho) + (\hat{\mu}+\xi)\hat{y}\left(v'(\hat{y}) - \phi\right),\end{aligned}$$

where the inequality follows from (A15). Now because any equilibrium must be at least as good as no trade, $v(y) \geq v^0(y) > \phi y - \rho$, and so the above implies

$$(\hat{\mu}+\xi)\left(v'(\hat{y}) - \phi\right) > 0.$$

From Step C.4, $v'(y) \leq \phi$, and therefore (A17) implies an immediate contradiction if $\hat{\mu}+\xi > 0$. We have not restricted the sign of $\hat{\mu}+\xi$, however, and so we next argue that at a singular point, $v'(\hat{y}) \geq \phi$. Combined with Step C.4, it follows that $v'(\hat{y}) = \phi$, and thus (A17) cannot be satisfied.

There are three cases to consider.

Case (i): The firm immediately issues debt $F > 0$.

In this case, $v(y) = v(\hat{y}) + v'(\hat{y})(y - \hat{y})$ for $y > \hat{y}$. Thus,

$$0 \leq v(y) - v^0(y) < v(y) - (\phi y - \rho) = (v'(\hat{y}) - \phi)y + \left(v(\hat{y}) - v'(\hat{y})\hat{y} - \rho\right).$$

Letting $y \to \infty$, the above implies $v'(\hat{y}) \geq \phi$.

Case (ii): Interior jump.

Suppose there is an interior interval $[\hat{y}, y_1]$ on which y is linear (and strictly convex at each end). Taking the difference between the HJB equations at $\hat{y}$ and y_1,

$$(r+\xi)v(y_1) = (1-\bar{\pi})(y_1 - c) - \xi + (\hat{\mu}+\xi)y_1 v'(y_1) + \frac{1}{2}\sigma^2 y_1^2 v''_+(y_1) + \lambda(v(\theta y_1) - v(y_1))$$

$$(r+\xi)v(\hat{y}) = (1-\bar{\pi})(\hat{y} - c) - \xi + (\hat{\mu}+\xi)\hat{y}v'(\hat{y}) + \frac{1}{2}\sigma^2 \hat{y}^2 v''_-(\hat{y}) + \lambda(v(\theta\hat{y}) - v(\hat{y}))$$

and using the fact that $v(y_1) = v(\hat{y}) + v'(\hat{y})(y_1 - \hat{y})$ and $v''_+(y_1) \geq 0 = v''_-(\hat{y})$, we have

$$\Big(r - \underbrace{\hat{\mu} + \lambda(\theta - 1)}_{\mu}\Big) v'(\hat{y})(y_1 - \hat{y}) \geq (1-\bar{\pi})(y_1 - \hat{y}) + \lambda\big(v(\theta y_1) - v(\theta\hat{y}) - \theta v'(\hat{y})(y_1 - \hat{y})\big)$$
$$\geq (1-\bar{\pi})(y_1 - \hat{y}),$$

where the second inequality follows from (A16). Dividing both sides by $(r - \mu)(y_1 - \hat{y})$, we have $v'(\hat{y}) \geq \phi$.

Case (iii): Isolated singular point.

From Step C.3, we know that $v''(\hat{y}) = 0$. Since it is isolated, the HJB holds on both sides of $\hat{y}$. By taking the derivative of the HJB and looking at the *right*-hand side of $\hat{y}$, since $v''(y) \geq 0$ at $\hat{y}+$, we have $v'''_+(\hat{y}) \geq 0$ and hence

$$(r+\xi)v'(\hat{y}) = 1 - \bar{\pi} + (\hat{\mu}+\xi)\Big[\hat{y}\underbrace{v''(\hat{y})}_{=0} + v'(\hat{y})\Big] + \tfrac{1}{2}\sigma^2\Big[\hat{y}^2\underbrace{v'''_+(\hat{y})}_{\geq 0} + 2\hat{y}\underbrace{v''(\hat{y})}_{=0}\Big]$$
$$+\lambda\Big(\underbrace{\theta v'(\theta\hat{y})}_{\geq \theta v'(\hat{y}),\, \text{as}\, \theta \geq 1} - v'(\hat{y})\Big)$$
$$\geq 1 - \bar{\pi} + \Big(\underbrace{\hat{\mu} + \lambda(\theta - 1)}_{\mu} + \xi\Big)v'(\hat{y}),$$

which implies that $v'(\hat{y}) \geq \phi$. This completes the proof of uniqueness. □

D. Endogenous Investment

We begin with a general proof that with a constant marginal tax rate, the no-trade value function is convex in F.

LEMMA D.1: (Convexity of No-Trade Value Function with Investment)*: Suppose the tax rate is constant and let $\hat{Y}_t^i$ be any after-tax cash flow process given some investment policy $i_t = i(Y_t, F_t)$. Then the no-trade value function is convex in F.*

PROOF: Given alternative initial debt levels (F_0, F_1) with maturity ξ, define

$$F_s^\alpha \equiv \alpha F_s{}^1 + (1-\alpha)F_s{}^0 = \alpha F_1 e^{-\xi s} + (1-\alpha)F_0 e^{-\xi s}.$$

Let $\hat{c} = c(1-\bar{\pi}) + \xi$ be the total after-tax debt burden per unit debt face value, and let $\hat{Y}_t^i$ be the after-tax cash flow process given investment policy $i_t = i(Y, F)$, so that the firm generates $\hat{Y}_t^i dt$ over the period $(t, t+dt)$. Then the no-trade value function, given the optimal investment $i(Y, F)$ and default $\tau(Y, F)$ policies, satisfies

$$\begin{aligned}
V(Y, \alpha F_1 + (1-\alpha)F_0) &= \max_{i,\tau} E\left[\int_0^\tau e^{-r(s-t)}\left(\hat{Y}_s^i - \hat{c}F_s{}^\alpha\right)ds\right] \\
&= \max_{i,\tau} E\left[\int_0^\tau e^{-r(s-t)}\left(\hat{Y}_s^i - \hat{c}\left[\alpha F_s{}^1 + (1-\alpha)F_s{}^0\right]\right)ds\right] \\
&= \max_{i,\tau} \alpha E\left[\int_0^\tau e^{-r(s-t)}\left(\hat{Y}_s^i - \hat{c}F_s{}^1\right)ds\right] + (1-\alpha)E\left[\int_0^\tau e^{-r(s-t)}\left(\hat{Y}_s^i - \hat{c}F_s{}^0\right)ds\right] \\
&\leq \alpha \max_{i,\tau} E\left[\int_0^\tau e^{-r(s-t)}\left(\hat{Y}_s^i - \hat{c}F_s{}^1\right)ds\right] + (1-\alpha)\max_{i,\tau} E\left[\int_0^\tau e^{-r(s-t)}\left(\hat{Y}_s^i - \hat{c}F_s{}^0\right)ds\right] \\
&= \alpha V(Y, F_1) + (1-\alpha)V(Y, F_0),
\end{aligned}$$

establishing the convexity of the no-trade value function with investment. □

With an endogenous investment choice $i_t \in [\mu_0, \mu]$, the firm's profitability (EBIT) evolves according to $dY_t = i_t Y_t dt + \sigma Y_t dZ_t$. The firm generates free cash flows of $\hat{Y}_t^i dt = (1-\bar{\pi})Y_t dt + \kappa(\mu - i_t)Y_t dt$ over the period $(t, t+dt)$, and the normalized no-trade value function $\hat{v}(\cdot)$ satisfies the HJB,

$$(r+\xi)\hat{v}(y) = \max_{i\in[\mu_0,\mu]} (1-\bar{\pi})(y-c) - \xi + \kappa(\mu - i)y + (i+\xi)y\hat{v}'(y) + \frac{1}{2}\sigma^2 y^2 \hat{v}''(y).$$

Investment is therefore optimal if and only if $\hat{v}'(y) \geq \kappa$. Convexity of the value function together with $\kappa < \phi$ therefore implies that it is optimal to invest whenever y exceeds some threshold y_i^{inv}. Because at default $\hat{v}'(y_b^{inv}) = 0 < \kappa$, the option to stop investing is valuable and hence $y_b^{inv} < y_b$.

To characterize $\hat{v}$, it is easiest to consider separately the value functions (v^I, v^{NI}) in the investment and no investment regions, respectively. The value function v^I can be solved using the methods of Section II; here, shareholders have the option to stop investing rather than to default.

Recall that $\phi \equiv \frac{1-\bar{\pi}}{r-\mu}$ and $\gamma \equiv \frac{(\mu+\xi-0.5\sigma^2)+\sqrt{(\mu+\xi-0.5\sigma^2)^2+2\sigma^2(r+\xi)}}{\sigma^2} > 0$. Given the value function $v^{NI}(\cdot)$ with $v^{NI}(y_b) > 0$ (because $y_b^{inv} < y_b$),

$$
\begin{aligned}
v^I(y) &= \max_{\hat{y}_i} \ \phi y - \rho + \left(\frac{y}{\hat{y}_i}\right)^{-\gamma} \left(v^{NI}(\hat{y}_i) - (\phi\hat{y}_i - \rho)\right) \\
&= \phi y - \rho + y^{-\gamma}\left[\max_{\hat{y}_i} \hat{y}_i^{\gamma}\left(v^{NI}(\hat{y}_i) - \phi\hat{y}_i + \rho\right)\right] \\
&> \phi y - \rho + y^{-\gamma}\left[y_b^{\gamma}\left(-\phi y_b + \rho\right)\right] = v(y),
\end{aligned}
$$

where $v(y)$ is the value function from Section II without the investment option. Therefore,

$$
v^I(y) = \ \phi y - \rho - \Lambda_I \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho)
$$

with

$$
\Lambda_I = \frac{\max_{\hat{y}_i} \hat{y}_i^{\gamma}\left(v^{NI}(\hat{y}_i) - \phi\hat{y}_i + \rho\right)}{y_b^{\gamma}\left(\rho - \phi y_b\right)} > 1.
$$

Note that $\Lambda_I > 1$ implies $v'(y) > v^{I'}(y) \geq \kappa > 0$ for $y \geq y_i^{inv}$. Because $v'(y_b) = 0$, we have $y_i^{inv} > y_b$.

We now solve for v^{NI} explicitly as follows. The HJB in the no-investment region implies

$$
v^{NI}(y) = \phi_d y - \rho + A_d y^{-\gamma_d} + B_d y^{\eta_d},
$$

where $\phi_d \equiv \frac{1-\bar{\pi}+\kappa(\mu-\mu_0)}{r}$, and $\eta_d \equiv \gamma_d + 1 - 2\xi\sigma^{-2} > 1$ with

$$
\gamma_d \equiv \frac{\left(\mu_0 + \xi - 0.5\sigma^2\right) + \sqrt{\left(\mu_0 + \xi - 0.5\sigma^2\right)^2 + 2\sigma^2\left(r + \xi\right)}}{\sigma^2} > 0.
$$

To solve the model, note that smooth-pasting and the optimality condition for investment imply

$$
v^I(y_i^{inv}) = v^{NI}(y_i^{inv}) \text{ and } v^{I'}\left(y_i^{inv}\right) = v^{NI'}\left(y_i^{inv}\right) = \kappa.
$$

Combined with the usual default boundary conditions,

$$
v^{NI}(y_b^{inv}) = v^{NI'}(y_b^{inv}) = 0,
$$

we have five equations to solve for five unknowns $\{\Lambda_I, A_d, B_d, y_i^{inv}, y_b^{inv}\}$. The solution to this system can be characterized as follows. The default boundary y_b^{inv} can be solved from the unidimensional nonlinear equation,

$$
\left[\frac{\left(\rho\gamma_d - \phi_d y_b^{inv}\left(1+\gamma_d\right)\right)\left(y_b^{inv}\right)^{-\eta_d}}{(1+\gamma_d)(\phi-\phi_d) + (\gamma_d/\gamma - 1)(\phi-\kappa)}\right]^{\frac{1}{1-\eta_d}} = \left[\frac{\left(\rho\eta_d - \phi_d y_b^{inv}\left(\eta_d - 1\right)\right)\left(y_b^{inv}\right)^{\gamma_d}}{(\eta_d - 1)(\phi-\phi_d) + (\eta_d/\gamma + 1)(\phi-\kappa)}\right]^{\frac{1}{1+\gamma_d}}.
$$

Once we obtain y_b^{inv}, the remaining unknowns are

$$y_i^{inv} = \left(\frac{\left(\rho\eta_d - \phi_d y_b^{inv}(\eta_d - 1)\right)\left(y_b^{inv}\right)^{\gamma_d}}{(\eta_d - 1)(\phi - \phi_d) + (\eta_d/\gamma + 1)(\phi - \kappa)} \right)^{\frac{1}{1+\gamma_d}}, \quad \Lambda_I = \left(1 - \frac{\kappa}{\phi}\right)\left(\frac{y_i^{inv}}{y_b}\right)^{1+\gamma},$$

$$A_d = -\frac{\phi_d y_b^{inv}(\eta_d - 1) - \rho\eta_d}{\gamma_d + \eta_d}\left(y_b^{inv}\right)^{\gamma_d}, \quad B_d = -\frac{\phi_d y_b^{inv}(1 + \gamma_d) - \rho\gamma_d}{\gamma_d + \eta_d}\left(y_b^{inv}\right)^{-\eta_d}.$$

Given the solution for (v^I, v^{NI}), we can then calculate the equilibrium debt price $p = yv' - v$ and issuance rate $g^* = \frac{\bar{\pi}c}{y^2 v''}$ as in the baseline model.

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